# DUALIZED SIMPLE TYPE THEORY

HARLEY EADES III, AARON STUMP, AND RYAN MCCLEEARY

Augusta University, 2500 Walton Way, Augusta, GA 30904
*e-mail address*: heades@augusta.edu

University of Iowa, 14 Maclean Hall, Iowa City, IA 52242-1419
*e-mail address*: aaron-stump@uiowa.edu

University of Iowa, 14 Maclean Hall, Iowa City, IA 52242-1419
*e-mail address*: ryan-mcleeary@uiowa.edu

Abstract. We propose a new bi-intuitionistic type theory called Dualized Type Theory (DTT). It is a simple type theory with perfect intuitionistic duality, and corresponds to a single-sided polarized sequent calculus. We prove DTT strongly normalizing, and prove type preservation. DTT is based on a new propositional bi-intuitionistic logic called Dualized Intuitionistic Logic (DIL) that builds on Pinto and Uustalu's logic L. DIL is a simplification of L by removing several admissible inference rules while maintaining consistency and completeness. Furthermore, DIL is defined using a dualized syntax by labeling formulas and logical connectives with polarities thus reducing the number of inference rules needed to define the logic. We give a direct proof of consistency, but prove completeness by reduction to L.

## 1. Introduction

The verification of software often requires the mixture of finite and infinite data types. The former are used to define tree-based structures while the latter are used to define infinite stream-based structures. An example of a tree-based structure is a list or an AVL tree. Infinite stream-based structures can be used to verify properties of a software system over time or to verify liveness properties of the system; see the introduction to [18] for a great discussion of the use of co-induction to study software systems. An example of an infinite stream-based structure is an infinitely branching tree, or an infinite list.

Finite tree-based structures can be modeled by inductive data types while infinite stream-based structures can be modeled by coinductive data types. Thus, tool support for reasoning about the behavior of a software system must provide both inductive data types as well as coinductive data types, and allow for their mixture. However, there are problems with existing systems that do provide both inductive and coinductive data types. For example, Agda restricts how inductive and coinductive types can be nested (see the discussion in [1]), while Coq supports general mixed inductive and coinductive data, but in doing so, sacrifices type preservation. Therefore, what is the proper logical foundation to study the relationships between inductive and coinductive data types? By studying such a

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foundation we may determine in what ways inductive and coinductive data can be mixed without sacrificing expressivity or key meta-theoretic properties.

One fairly obvious relationship between inductive and coinductive data types is that they are duals to each other. We believe that the proper foundation for studying inductive and coinductive types must be able to express this symmetry while maintaining constructivity. It turns out that a constructive logical foundation may lie in an already known constructive logic known as bi-intuitionistic logic.

Bi-intuitionistic logic (BINT)[1] is an conservative extension [8] of intuitionistic logic with prefect duality. That is, every logical connective in the logic has a dual. For example, BINT contains conjunction, and disjunction, their units true and false, but also implication and its dual called co-implication (also known as subtraction, difference, or exclusion).

Co-implication is fairly unknown in computer science, but an intuition of its meaning can be seen in its interpretation into Kripke models. In [29, 30] Rauszer gives a conservative extension of the Kripke semantics for intuitionistic logic that models all of the logical connectives of BINT by introducing a new logical connective for co-implication. The usual interpretation of implication in a Kripke model is as follows:

$$[\![A \to B]\!]_w = \forall w'.w \leq w' \to [\![A]\!]_{w'} \to [\![B]\!]_{w'}$$

Rauszer took the dual of the previous interpretation to obtain the following:

$$[\![A - B]\!]_w = \exists w'.w' \leq w \land \neg[\![A]\!]_{w'} \land [\![B]\!]_{w'}$$

The previous interpretation shows that implication considers future worlds, while co-implication considers past worlds.

We consider BINT logic to be the closest extension of intuitionistic logic to classical logic while maintaining constructivity. BINT has two forms of negation, one defined as usual, $\neg A \stackrel{\mathsf{def}}{=} A \to\perp$, and a second defined in terms of co-implication, $\sim\!A \stackrel{\mathsf{def}}{=} \top - A$. The latter we call "non-$A$". Now in BINT it is possible to prove $A \lor \sim\!A$ for any $A$ [7]. In fact, the latter, in a type theoretic setting, corresponds to the type of a constructive control operator [8].

BINT is a conservative extension of intuitionistic logic, and hence maintains constructivity, but contains a rich notion of symmetry between the logical connectives. Thus, any extension of a BINT logic must preserve this symmetry, and hence, if we add inductive data types, then we must also add co-inductive data types. However, all of this is premised on the ability to define a BINT type theory.

The contributions of this paper are a new formulation of Pinto and Uustalu's BINT labeled sequent calculus L called Dualized Intuitionistic Logic (DIL) and a corresponding type theory called Dualized Type Theory (DTT). DIL is a single-sided polarized formulation of Pinto and Uustalu's L, thus, DIL is a propositional bi-intuitionistic logic, and builds on L by removing the following rules (see Section 3 for a complete definition of L):

[1]We only consider propositional logic in this paper. Note that first-order BINT is non-conservative over first-order intuitionistic logic [30, 15], but we believe that second-order BINT is conservative over second-order intuitionistic logic, but we leave this to future work.

$$\frac{\Gamma \vdash_{G\cup\{(n,n)\}} \Delta}{\Gamma \vdash_G \Delta}\ \textsc{refl} \qquad \frac{\begin{array}{l} n_1 G n_2 \\ n_2 G n_3 \\ \Gamma \vdash_{G\cup\{(n_1,n_3)\}} \Delta \end{array}}{\Gamma \vdash_G \Delta}\ \textsc{trans}$$

$$\frac{\begin{array}{l} nGn' \\ \Gamma, n : T, n' : T \vdash_G \Delta \end{array}}{\Gamma, n : T \vdash_G \Delta}\ \textsc{mon}\mathrm{L} \qquad \frac{\begin{array}{l} n'Gn \\ \Gamma \vdash_G n' : T, n : T, \Delta \end{array}}{\Gamma \vdash_G n : T, \Delta}\ \textsc{mon}\mathrm{R}$$

We show that in the absence of the previous rules DIL still maintains consistency (Theorem 12) and completeness (Theorem 43). Furthermore, DIL is defined using a dualized syntax which reduces the number of inference rules needed to define the logic.

Since DIL has multiple conclusions, and the active formula is on the right, DIL must have a means of switching out the active formula with another conclusion. This is done in DIL using cuts on hypotheses. We call these types of cuts "axiom cuts." These axiom cuts show up in non-trivial proofs like the proof of the axiom $A \vee {\sim}A$ for any $A$ [7]. Furthermore, when the latter is treated as a type in DTT, the inhabitant is a continuation without a canonical form, because the inhabitant contains as a subexpression an axiom cut. Thus, the presence of these continuations prevents the canonicity result for a type theory – like DTT – from holding. Thus, if general cut elimination was a theorem of DIL, then $A \vee {\sim}A$ would not be provable. So DIL must contain cuts that cannot be eliminated. This implies that DIL does not enjoy general cut elimination, but all cuts other than axiom cuts can be eliminated. Throughout the sequel we define "cut elimination" as the elimination of all cuts other than axiom cuts, and we call DIL "cut free" with respect to this definition of cut elimination. The latter point is similar to Wadler's dual calculus [36].

The general form of a DIL sequent is $G; \Gamma \vdash p\, A \,@\, n$ where $\Gamma$ is a context, multiset of hypotheses of the form $p'\, B \,@\, n'$, $p$ is a polarity which can be either $+$ or $-$, and $n$ is a node of the abstract Kripke graph $G$ which is a list of edges. Think of $G$ as a list of constraints on the accessibility relation in the Kripke semantics. The negative hypotheses in $\Gamma$ are alternate conclusions. In fact, if we denote by $\Gamma^p$ the subcontext of $\Gamma$ consisting of all the hypotheses with polarity $p$, then we can translate a DIL sequent, $G; \Gamma \vdash p\, A \,@\, n$, into the more traditional form where if $p = +$, then the sequent is equivalent to $G; \Gamma^+ \vdash +\, A \,@\, n, \Gamma^-$, but if $p = -$, then the sequent is equivalent to $G; \Gamma^+, -\, A \,@\, n \vdash \Gamma^-$.

The polarities provide two main properties of DIL and DTT. The first, which is more fundamental than the second, is the ability to single out an active formula providing a single-conclusion perspective of a multi-conclusion logic. This is important if we want to obtain a type theory in the traditional form: a single term on the right. The second main property is they provide a means of significantly reducing the number of inference rules that define the logic. Above we saw that in $G; \Gamma \vdash +\, A \,@\, n$ we think of $A$ as being on the right, but in $G; \Gamma \vdash -\, A \,@\, n$ we think of $A$ as being on the left, and thus, if we index the logical operators of DIL with polarities, for example in $A \wedge_p B$, we can collapse the left and right rules into a single rule. For example, $A \wedge_+ B$ is conjunction, but $A \wedge_- B$ is disjunction, and the right-rule for conjunction mirrors the left-rule for disjunction, but we move from the right to the left, but in DIL this is just a change in polarity. The right-rule for conjunction and the left-rule for disjunction can thus be given by the single rule:

$$\frac{G; \Gamma \vdash p\, A \,@\, n \qquad G; \Gamma \vdash p\, B \,@\, n}{G; \Gamma \vdash p\, (A \wedge_p B) \,@\, n}\ \textsc{and}$$

A summary of our contributions is as follows:

- A new formulation of Pinto and Uustalu's BINT labeled sequent calculus L called Dualized Intuitionistic Logic (DIL),
- a corresponding simple type theory called Dualized Type Theory (DTT),
- a computer-checked proof – in Agda – of consistency for DIL with respect to Rauszer's Kripke semantics for BINT logic,
- a completeness proof for DIL by reduction to Pinto and Uustalu's L, and
- the basic metatheory for DTT: type preservation and strong normalization for DTT. We show the latter using a version of Krivine's classical realizability by translating DIL into a classical logic.

The rest of this paper is organized as follows. We first discuss related work in Section 2. Then we introduce Pinto and Uustalu's L calculus in Section 3, and then DIL in Section 4. We present the consistency proof for DIL in Section 4.1, and then show DIL is complete (with only axiom cuts) in Section 4.2. Following DIL we introduce DTT in Section 5, and its metatheory in Section 6. All of the mathematical content of this paper was typeset with the help of Ott [33].

## 2. Related Work

The main motivation for studying BINT is to use it to study the mixture of inductive and co-inductive data types, but from a constructive perspective. However, a natural question to ask is can classical logic be used? There has been a lot of work done since Griffin's seminal paper [14] showing that the type of Peirce's law corresponds to a control operator, and thus, providing a means of defining a program from any classical proof; for example see [27, 31, 9, 36]. Kimura and Tatsuta extend Wadler's Dual Calculus (DC) with inductive and coinductive data types in [19]. The Dual Calculus was invented by Wadler [36], and is a multi-conclusion classical simple type theory based in sequent calculus instead of natural deduction. DC only contains the logical operators conjunction, disjunction, and negation. Then he defines the other operators in terms of these. Thus, co-implication is defined, and not taken as a primitive operator. Kimura and Tatsuta carry out a very similar program to what we are proposing here. They add inductive and co-inductive types to DC, show that the rich symmetry of classical logic extends to inductive and co-inductive types, and finally shows how to embed this extension into the second-order extension of DC. The starkest difference between their work, and the ultimate goals of our program is that we wish to be as constructive as possible. We choose to do this, because we ultimately wish to extend our work to dependent types which we conjecture will be a goal more easily reached in a constructive setting versus a classical setting. Extending control operators to dependent types is currently an open problem; for example, general $\Sigma$-types cannot be mixed with control operators [16].

As we mentioned above BINT logic is fairly unknown in computer science. Crolard introduced a logic and corresponding type theory called subtractive logic, and showed it can be used to study constructive coroutines in [7, 8]. He initially defined subtractive logic in sequent style with the Dragalin restriction, and then defined the corresponding type theory in natural deduction style by imposing a restriction on Parigot's $\lambda\mu$-calculus in the form of complex dependency tracking. Just as linear logicians have found – for example in [32] – Pinto and Uustalu were able to show that imposing the Dragalin restriction in subtractive logic results in a failure of cut elimination [28]. They recover cut elimination by proposing a new BINT logic called L that lifts the Dragalin restriction by labeling formulas and sequents with nodes and graphs respectively; this labeling corresponds to placing constraints on the sequents where the graphs can be seen as abstract Kripke models. Goré et al. also proposed a new BINT logic that enjoys cut elimination using nested sequents; however it is currently unclear how to define a type theory with nested sequents [13]. Bilinear logic in its intuitionistic form is a linear version of BINT and has been studied by Lambek in [21, 22]. Biasi and Aschieri propose a term assignment to polarized bi-intuitionistic logic in [6]. One can view the polarities of their logic as an internalization of the polarities of the logic we propose in this article. Bellin has studied BINT similar to that of Biasi and Aschieri from a philosophical perspective in [2, 3, 4], and he defined a linear version of Crolard's subtractive logic for which he was able to construct a categorical model using linear categories in [5].

DIL sequents are labeled with an abstract Kripke graph which is defined as a multiset of edges between abstract nodes – labels denoted $n$. Then all formulas in a sequent are labeled with a node from the graph, and the inference rules of DIL are restricted using conditions on the graph and the nodes on formulas which are based on the interpretation of formula into the Kripke semantics. This idea in BINT logic comes from Pinto and Uustalu's L [28], but their work was inspired by Negri's work on contraction and cut-free modal logics [25].

A system related to both L and DIL is Reed and Pfenning's labeled intuitionistic logic with a restricted notion of control operators. Their logic can also be seen as a restriction of classical logic by labeling the formulas with strings of nodes representing a directed path in the Kripke semantics. That is, a formula is of the form $A[p]$ where $p$ is a string of nodes where if $p = n_1 n_2 \cdots n_{i-1} n_i$ then we can intuitively think of $p$ as a path in the Kripke semantics, and hence, $p$ represents the path $R\,n_1\,(R\,n_2\,(\cdots(R\,n_{i-1}\,n_i)\cdots))$, where $R$ is the accessibility relation. One very interesting aspect of their natural deduction formulation – which has a term assignment – is that it contains the terms $\mathsf{throw}$ and $\mathsf{catch}$ which are used to allow for multiple conclusions. These give the logic some control like operators intuitionisticly. We conjecture that the propositional fragment of Reed and Pfenning's system should be able to be embedded into DIL fairly straightforwardly. In fact, $\mathsf{throw}$ and $\mathsf{catch}$ correspond to our axiom cuts mentioned in Section 4.2 which allows DIL to switch between the multiple conclusions. Both L and DIL have a more general labeling than Reed and Pfenning's system, because theirs only speaks about a single path, and future worlds along that path, but L and DIL allow one to talk about multiple different paths, and consider both future and past worlds.

Similarly, to L, DIL, DTT, and Reed and Pfenning's logic Murphy et al. use a labeling system that annotates formulas with worlds, and use world constraints to restrict the logic [24]. They even use the same syntax as DIL and DTT, that is, their formulas are denoted by $A@w$ which stands for $A$ is true at the world $w$. It would be interesting to see if their work provides a means of extending DIL and DTT to BINT modal logic.

An alternative approach to Pinto and Uustalu's L was given by Galmiche and Méry [11]. They give a labeled sequent calculus for BINT and a counter-model construction similarly to L, but they use a different method for constructing the labeled sequent calculus called connection-based validity. Their systems uses a different notion of graph called R-graphs that annotate sequents, but these graphs are far more complex than the abstract Kripke graphs of DIL and L. Méry et al. later implement an interactive theorem prover for their system [23].

$$
\begin{array}{rrcl}
\text{(formulas)} & A, B, C & ::= & \top \mid \perp \mid A \supset B \mid A \prec B \mid A \wedge B \mid A \vee B \\
\text{(graphs)} & G & ::= & \cdot \mid (n, n') \mid G, G' \\
\text{(contexts)} & \Gamma, \Delta & ::= & \cdot \mid n : A \mid \Gamma, \Gamma' \\
\text{(sequents)} & Q & ::= & \Gamma \vdash_G \Delta
\end{array}
$$

Figure 1: Syntax of L.

## 3. Pinto and Uustalu's L

In this section we briefly introduce Pinto and Uustalu's L from [28]. The syntax for formulas, graphs, and contexts of L are defined in Figure 1, while the inference rules are defined in Figure 2. The formulas include true and false denoted $\top$ and $\perp$ respectively, implication and co-implication denoted $A \supset B$ and $A \prec B$ respectively, and finally, conjunction and disjunction denoted $A \wedge B$ and $A \vee B$ respectively. So we can see that for every logical connective its dual is a logical connective of the logic. This is what we meant by BINT containing perfect intuitionistic duality in the introduction. Sequents have the form $\Gamma \vdash_G \Delta$, where $\Gamma$ and $\Delta$ are multisets of formulas $n : A$ labeled by a node $n$, $G$ is the abstract Kripke model or sometimes referred to as simply the graph of the sequent, and $n$ is a node in $G$.

A graph is a multiset of directed edges where each edge is a pair of nodes. One should view these edges as constraints on the accessibility relation in the Kripke semantics; see the interpretation of graphs in Definition 7 and the definition validity for L in Definition 41. We denote $(n_1, n_2) \in G$ by $n_1 G n_2$. Furthermore, we denote the union of two graphs $G$ and $G'$ as $G \cup G'$. Now each formula present in a sequent is labeled with a node in the graph. This labeling is denoted $n : A$ and should be read as the formula $A$ is true at the node $n$. We denote the operation of constructing the list of nodes in a graph or context by $|G|$ and $|\Gamma|$ respectively. The reader should note that it is possible for some nodes in the sequent to not appear in the graph. For example, the sequent $n : A \vdash_. n : A, \cdot$ is a derivable sequent. The complete graph can always be recovered if needed by using the graph structural rules REFL, TRANS, MONL, and MONR.

Consistency of L is stated in [28] without a detailed proof, but is proven complete with respect to Rauszer's Kripke semantics using a counter model construction. In Section 4 we give a translation of the formulas of L into the formulas of DIL (Section 4.2.1) and a translation in the inverse direction (Section 4.2.2), which are both used to show completeness of DIL in Section 4.2.

## 4. Dualized Intuitionistic Logic (DIL)

The syntax for polarities, formulas, and graphs of DIL is defined in Figure 3, where **a** ranges over atomic formulas. The following definition shows that DIL's formulas are simply polarized versions of L's formulas.

**Definition 1.** The following defines a translation of formulas of L to formulas of DIL:

$$
\begin{array}{rclcrclcrcl}
\mathsf{D}(\top) & = & \langle + \rangle & & \mathsf{D}(A \wedge B) & = & \mathsf{D}(A) \wedge_+ \mathsf{D}(B) & & \mathsf{D}(A \supset B) & = & \mathsf{D}(A) \to_+ \mathsf{D}(B) \\
\mathsf{D}(\perp) & = & \langle - \rangle & & \mathsf{D}(A \vee B) & = & \mathsf{D}(A) \wedge_- \mathsf{D}(B) & & \mathsf{D}(B \prec A) & = & \mathsf{D}(A) \to_- \mathsf{D}(B)
\end{array}
$$

We represent graphs as lists of edges denoted $n_1 \preccurlyeq_p n_2$, where we denote an edge from $n_1$ to $n_2$ by $n_1 \preccurlyeq_+ n_2$, and we denote the edge from $n_2$ to $n_1$ by $n_1 \preccurlyeq_- n_2$. Lastly, contexts denoted $\Gamma$ are represented as lists of formulas. Throughout the sequel we denote

$$\frac{\Gamma \vdash_{G\cup\{(n,n)\}} \Delta}{\Gamma \vdash_G \Delta}\ \textsc{refl} \qquad \frac{\begin{array}{l} n_1 G n_2 \\ n_2 G n_3 \\ \Gamma \vdash_{G\cup\{(n_1,n_3)\}} \Delta \end{array}}{\Gamma \vdash_G \Delta}\ \textsc{trans} \qquad \frac{}{\Gamma, n : T \vdash_G n : T, \Delta}\ \textsc{hyp}$$

$$\frac{\begin{array}{l} nGn' \\ \Gamma, n : T, n' : T \vdash_G \Delta \end{array}}{\Gamma, n : T \vdash_G \Delta}\ \textsc{mon}\mathrm{L} \qquad \frac{\begin{array}{l} n'Gn \\ \Gamma \vdash_G n' : T, n : T, \Delta \end{array}}{\Gamma \vdash_G n : T, \Delta}\ \textsc{mon}\mathrm{R}$$

$$\frac{\Gamma \vdash_G \Delta}{\Gamma, n : \top \vdash_G \Delta}\ \textsc{true}\mathrm{L} \qquad \frac{}{\Gamma \vdash_G n : \top, \Delta}\ \textsc{true}\mathrm{R} \qquad \frac{}{\Gamma, n : \perp \vdash_G \Delta}\ \textsc{false}\mathrm{L}$$

$$\frac{\Gamma \vdash_G \Delta}{\Gamma \vdash_G n : \perp, \Delta}\ \textsc{false}\mathrm{R} \qquad \frac{\Gamma, n : T_1, n : T_2 \vdash_G \Delta}{\Gamma, n : T_1 \wedge T_2 \vdash_G \Delta}\ \textsc{and}\mathrm{L}$$

$$\frac{\begin{array}{c} \Gamma \vdash_G n : T_1, \Delta \\ \Gamma \vdash_G n : T_2, \Delta \end{array}}{\Gamma \vdash_G n : T_1 \wedge T_2, \Delta}\ \textsc{and}\mathrm{R} \qquad \frac{\begin{array}{c} \Gamma, n : T_1 \vdash_G \Delta \\ \Gamma, n : T_2 \vdash_G \Delta \end{array}}{\Gamma, n : T_1 \vee T_2 \vdash_G \Delta}\ \textsc{disj}\mathrm{L}$$

$$\frac{\Gamma \vdash_G n : T_1, n : T_2, \Delta}{\Gamma \vdash_G n : T_1 \vee T_2, \Delta}\ \textsc{disj}\mathrm{R} \qquad \frac{\begin{array}{l} nGn' \\ \Gamma \vdash_G n' : T_1, \Delta \\ \Gamma, n' : T_2 \vdash_G \Delta \end{array}}{\Gamma, n : T_1 \supset T_2 \vdash_G \Delta}\ \textsc{imp}\mathrm{L}$$

$$\frac{\begin{array}{l} n' \notin |G|, |\Gamma|, |\Delta| \\ \Gamma, n' : T_1 \vdash_{G\cup\{(n,n')\}} n' : T_2, \Delta \end{array}}{\Gamma \vdash_G n : T_1 \supset T_2, \Delta}\ \textsc{imp}\mathrm{R} \qquad \frac{\begin{array}{l} n' \notin |G|, |\Gamma|, |\Delta| \\ \Gamma, n' : T_1 \vdash_{G\cup\{(n,n')\}} n' : T_2, \Delta \end{array}}{\Gamma, n' : T_1 \prec T_2 \vdash_G \Delta}\ \textsc{sub}\mathrm{L}$$

$$\frac{\begin{array}{l} n'Gn \\ \Gamma \vdash_G n' : T_1, \Delta \\ \Gamma, n' : T_2 \vdash_G \Delta \end{array}}{\Gamma \vdash_G n : T_1 \prec T_2, \Delta}\ \textsc{sub}\mathrm{R}$$

Figure 2: Inference Rules for L.

$$\begin{array}{rrcl} \text{(polarities)} & p & ::= & + \mid - \\ \text{(formulas)} & A, B, C & ::= & \mathbf{a} \mid \langle p \rangle \mid A \to_p B \mid A \wedge_p B \\ \text{(graphs)} & G & ::= & \cdot \mid n \preccurlyeq_p n' \mid G, G' \\ \text{(contexts)} & \Gamma & ::= & \cdot \mid p\, A \mathbin{@} n \mid \Gamma, \Gamma' \\ \text{(sequents)} & Q & ::= & G; \Gamma \vdash p\, A \mathbin{@} n \end{array}$$

Figure 3: Syntax for DIL.

the opposite of a polarity $p$ by $\bar{p}$. This is defined by $\bar{+} = -$ and $\bar{-} = +$. The inference rules for DIL are in Figure 4.

$$\frac{G \vdash n \preccurlyeq_p^* n'}{G; \Gamma, p\, A \,@\, n, \Gamma' \vdash p\, A \,@\, n'}\ \textsc{ax} \qquad \frac{}{G; \Gamma \vdash p\, \langle p \rangle \,@\, n}\ \textsc{unit}$$

$$\frac{G; \Gamma \vdash p\, A \,@\, n \qquad G; \Gamma \vdash p\, B \,@\, n}{G; \Gamma \vdash p\, (A \wedge_p B) \,@\, n}\ \textsc{and} \qquad \frac{G; \Gamma \vdash p\, A_d \,@\, n}{G; \Gamma \vdash p\, (A_1 \wedge_{\bar{p}} A_2) \,@\, n}\ \textsc{andBar}$$

$$\frac{\begin{array}{l} n' \notin |G|, |\Gamma| \\ (G, n \preccurlyeq_p n'); \Gamma, p\, A \,@\, n' \vdash p\, B \,@\, n' \end{array}}{G; \Gamma \vdash p\, (A \rightarrow_p B) \,@\, n}\ \textsc{imp}$$

$$\frac{\begin{array}{l} G \vdash n \preccurlyeq_{\bar{p}}^* n' \\ G; \Gamma \vdash \bar{p}\, A \,@\, n' \qquad G; \Gamma \vdash p\, B \,@\, n' \end{array}}{G; \Gamma \vdash p\, (A \rightarrow_{\bar{p}} B) \,@\, n}\ \textsc{impBar}$$

$$\frac{G; \Gamma, \bar{p}\, A \,@\, n \vdash +\, B \,@\, n' \qquad G; \Gamma, \bar{p}\, A \,@\, n \vdash -\, B \,@\, n'}{G; \Gamma \vdash p\, A \,@\, n}\ \textsc{cut}$$

Figure 4: Inference Rules for DIL.

$$\frac{}{G, n \preccurlyeq_p n', G' \vdash n \preccurlyeq_p^* n'}\ \textsc{rel\_ax} \qquad \frac{}{G \vdash n \preccurlyeq_p^* n}\ \textsc{rel\_refl}$$

$$\frac{G \vdash n \preccurlyeq_p^* n' \qquad G \vdash n' \preccurlyeq_p^* n''}{G \vdash n \preccurlyeq_p^* n''}\ \textsc{rel\_trans} \qquad \frac{G \vdash n' \preccurlyeq_{\bar{p}}^* n}{G \vdash n \preccurlyeq_p^* n'}\ \textsc{rel\_flip}$$

Figure 5: Reachability Judgment for DIL.

The sequent has the form $G; \Gamma \vdash p\, A \,@\, n$ which when $p$ is positive (resp. negative) can be read as the formula $A$ is true (resp. false) at node $n$ in the context $\Gamma$ with respect to the graph $G$. Note that the metavariable $d$ in the premise of the AndBar rule ranges over the set $\{1, 2\}$ and prevents the need for two rules. The inference rules depend on a reachability judgment that provides a means of proving when a node is reachable from another within some graph $G$. This judgment is defined in Figure 5. In addition, the imp rule depends on the operations $|G|$ and $|\Gamma|$ which simply compute the list of all the nodes in $G$ and $\Gamma$ respectively. The condition $n' \notin |G|, |\Gamma|$ in the imp rule is required for consistency.

The most interesting inference rules of DIL are the rules for implication and co-implication from Figure 4. Let us consider these two rules in detail. These rules mimic the definitions of the interpretation of implication and co-implication in a Kripke model. The imp rule states that the formula $p\, (A \rightarrow_p B)$ holds at node $n$ if $p\, A \,@\, n'$ holds at an arbitrary node $n'$ where we add a new edge $n \preccurlyeq_p n'$ to the graph, then $p\, B \,@\, n'$ holds. Notice that when $p$ is positive $n'$ will be a future node, but when $p$ is negative $n'$ will be a past node. Thus, universally quantifying over past and future worlds is modeled here by adding edges to the graph. Now the impBar rule states the formula $p\, (A \rightarrow_{\bar{p}} B)$ is derivable if there exists a node $n'$ that is provably reachable from $n$, $\bar{p}\, A$ is derivable at node $n'$, and $p\, B \,@\, n'$ is derivable at node $n'$. When $p$ is positive $n'$ will be a past node, but when $p$ is negative

$n'$ will be a future node. This is exactly dual to implication. Thus, existence of past and future worlds is modeled by the reachability judgment.

Before moving on to proving consistency and completeness of DIL we first show that the formula $A \wedge_- \sim A$ has a proof in DIL that contains a cut that cannot be eliminated. This also serves as an example of a derivation in DIL. Consider the following where we leave off the reachability derivations for clarity and $\Gamma' \equiv -(A \wedge_- \sim A) @ n, -A @ n$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{G;\Gamma,\Gamma' \vdash -A @ n}\ \textsc{ax} \qquad \dfrac{}{G;\Gamma,\Gamma' \vdash +\langle + \rangle @ n}\ \textsc{unit}}{G;\Gamma,\Gamma' \vdash + \sim A @ n}\ \textsc{impBar}}{G;\Gamma,\Gamma' \vdash +(A \wedge_- \sim A) @ n}\ \textsc{andBar} \qquad \dfrac{}{G;\Gamma,\Gamma' \vdash -(A \wedge_- \sim A) @ n}\ \textsc{ax}}{\dfrac{G;\Gamma,-(A \wedge_- \sim A) @ n \vdash +A @ n}{G;\Gamma,-(A \wedge_- \sim A) @ n \vdash +(A \wedge_- \sim A) @ n}\ \textsc{andBar}}\ \textsc{cut}$$

Now using only an axiom cut we may conclude the following derivation:

$$\dfrac{G;\Gamma,-(A \wedge_- \sim A) @ n \vdash +(A \wedge_- \sim A) @ n \qquad \dfrac{}{G;\Gamma,-(A \wedge_- \sim A) @ n \vdash -(A \wedge_- \sim A) @ n}\ \text{AX}}{G;\Gamma \vdash +(A \wedge_- \sim A) @ n}\ \text{CUT}$$

The reader should take notice to the fact that all cuts within the previous two derivations are axiom cuts – see the introduction to Section 4.2 for the definition of axiom cuts – where the inner most cut uses the hypothesis of the outer cut. Therefore, neither can be eliminated.

4.1. **Consistency of DIL.** In this section we prove consistency of DIL with respect to Rauszer's Kripke semantics for BINT logic. All of the results in this section have been formalized in the Agda proof assistant[2]. We begin by first defining a Kripke frame.

**Definition 2.** A **Kripke frame** is a pair $(W, R)$ of a set of worlds $W$, and a preorder $R$ on $W$.

Then we extend the notion of a Kripke frame to include an evaluation for atomic formulas resulting in a Kripke model.

**Definition 3.** A **Kripke model** is a tuple $(W, R, V)$, such that, $(W, R)$ is a Kripke frame, and $V$ is a binary monotone relation on $W$ and the set of atomic formulas of DIL.

Now we can interpret formulas in a Kripke model as follows:

**Definition 4.** The interpretation of the formulas of DIL in a Kripke model $(W, R, V)$ is defined by recursion on the structure of the formula as follows:

$$\begin{array}{rcl} [\![\langle + \rangle]\!]_w & = & \top \\ [\![\langle - \rangle]\!]_w & = & \bot \\ [\![\mathbf{a}]\!]_w & = & V\,w\,\mathbf{a} \end{array} \qquad \begin{array}{rcl} [\![A \wedge_+ B]\!]_w & = & [\![A]\!]_w \wedge [\![B]\!]_w \\ [\![A \wedge_- B]\!]_w & = & [\![A]\!]_w \vee [\![B]\!]_w \\ [\![A \to_+ B]\!]_w & = & \forall w' \in W.R\,w\,w' \to [\![A]\!]_{w'} \to [\![B]\!]_{w'} \\ [\![A \to_- B]\!]_w & = & \exists w' \in W.R\,w'\,w \wedge \neg[\![A]\!]_{w'} \wedge [\![B]\!]_{w'} \end{array}$$

The interpretation of formulas really highlights the fact that implication is dual to co-implication. Monotonicity holds for this interpretation.

**Lemma 5** (Monotonicity)**.** Suppose $(W, R, V)$ is a Kripke model, $A$ is some DIL formula, and $w, w' \in W$. Then $R\,w\,w'$ and $[\![A]\!]_w$ imply $[\![A]\!]_{w'}$.

[2]Agda source code is available at `https://github.com/heades/DIL-consistency`

At this point we must set up the mathematical machinery which allows for the interpretation of sequents in a Kripke model. This will require the interpretation of graphs, and hence, nodes. We interpret nodes as worlds in the model using a function we call a node interpreter.

**Definition 6.** Suppose $(W, R, V)$ is a Kripke model and $S$ is a set of nodes of an abstract Kripke model $G$. Then a **node interpreter** on $S$ is a function from $S$ to $W$.

Now using the node interpreter we can interpret edges as statements about the reachability relation in the model. Thus, the interpretation of a graph is just the conjunction of the interpretation of its edges.

**Definition 7.** Suppose $(W, R, V)$ is a Kripke model, $G$ is an abstract Kripke model, and $N$ is a node interpreter on the set of nodes of $G$. Then the interpretation of $G$ in the Kripke model is defined by recursion on the structure of the graph as follows:

$$\begin{array}{lcl} [\![\cdot]\!]_N & = & \top \\ [\![n_1 \preccurlyeq_+ n_2, G]\!]_N & = & R\,(N\,n_1)\,(N\,n_2) \wedge [\![G]\!]_N \\ [\![n_1 \preccurlyeq_- n_2, G]\!]_N & = & R\,(N\,n_2)\,(N\,n_1) \wedge [\![G]\!]_N \end{array}$$

The reachability judgment of DIL provides a means to prove that two particular nodes are reachable in the abstract Kripke graph, but this proof is really just a syntactic proof of transitivity. The following lemma makes this precise.

**Lemma 8** (Reachability Interpretation)**.** Suppose $(W, R, V)$ is a Kripke model, and $[\![G]\!]_N$ for some abstract Kripke graph $G$. Then

i. if $G \vdash n_1 \preccurlyeq_+ n_2$, then $R\,(N\,n_1)\,(N\,n_2)$, and
ii. if $G \vdash n_1 \preccurlyeq_- n_2$, then $R\,(N\,n_2)\,(N\,n_1)$.

We have everything we need to interpret abstract Kripke models. The final ingredient to the interpretation of sequents is the interpretation of contexts.

**Definition 9.** If $F$ is some meta-logical formula, we define $p\,F$ as follows:

$$+\,F = F \quad \text{and} \quad -\,F = \neg F.$$

**Definition 10.** Suppose $(W, R, V)$ is a Kripke model, $\Gamma$ is a context, and $N$ is a node interpreter on the set of nodes in $\Gamma$. The interpretation of $\Gamma$ in the Kripke model is defined by recursion on the structure of the context as follows:

$$\begin{array}{lcl} [\![\cdot]\!]_N & = & \top \\ [\![p\ A \mathbin{@} n, \Gamma]\!]_N & = & p[\![A]\!]_{(N\,n)} \wedge [\![\Gamma]\!]_N \end{array}$$

Combining these interpretations results in the following definition of validity.

**Definition 11.** Suppose $(W, R, V)$ is a Kripke model, $\Gamma$ is a context, and $N$ is a node interpreter on the set of nodes in $\Gamma$. The interpretation of sequents is defined as follows:

$$[\![G; \Gamma \vdash p\ A \mathbin{@} n]\!]_N \quad = \quad \text{if } [\![G]\!]_N \text{ and } [\![\Gamma]\!]_N, \text{ then } p[\![A]\!]_{(N\,n)}.$$

Then a sequent $G; \Gamma \vdash p\ A \mathbin{@} n$ is valid when $[\![G; \Gamma \vdash p\ A \mathbin{@} n]\!]_N$ holds for any $N$ and in any Kripke model.

Notice that in the definition of validity the graph $G$ is interpreted as a set of constraints imposed on the set of Kripke models, thus reinforcing the fact that the graphs on sequents really are abstract Kripke models. Finally, using the previous definition of validity we can prove consistency.

**Theorem 12** (Consistency)**.** Suppose $G;\Gamma \vdash p\ A @ n$. Then for any Kripke model $(W, R, V)$ and node interpreter $N$ on $|G|$, $[\![G;\Gamma \vdash p\ A @ n]\!]_N$.

4.2. **Completeness of DIL.** DIL has a tight correspondence with Pinto and Uustalu's L. In [28] it is shown that L is complete with respect to Kripke models using a counter-model construction; see Corollary 1 on p. 13 of ibid. We will exploit their completeness result to show that DIL is complete. First, we will give a pair of translations: one from L to DIL (Definition 29), and one from DIL to L (Definition 32). Using these translations we will show that if a L-sequent[3] is derivable, then its translation to DIL is also derivable (Lemma 31), and vice versa (Lemma 37). Next we will relate validity of DIL with validity of L, and show that if a DIL-sequent is valid with respect to the semantics of DIL, then its translation to L is valid with respect to the semantics of L (Lemma 42). Finally, we can use the previous result to show completeness of DIL (Theorem 43).

Throughout this section we assume without loss of generality that all L-sequents have non-empty right-hand sides. That is, for every L-sequent, $\Gamma \vdash_G \Delta$, we assume that $\Delta \neq \cdot$. We do not loose generality because it is possible to prove that $\Gamma \vdash_G \cdot$ holds if and only if $\Gamma \vdash_G n :\perp$ for any node $n$ (proof omitted).

We proved DIL consistent when DIL contained the general cut rule, but we prove DIL complete when the cut rule has been replaced with the following two inference rules, which can be seen as restricted instances of the cut rule:

$$\frac{p\ B @ n' \in (\Gamma, \bar{p}\ A @ n) \qquad G;\Gamma, \bar{p}\ A @ n \vdash \bar{p}\ B @ n'}{G;\Gamma \vdash p\ A @ n}\ \textsc{axCut}$$

$$\frac{\bar{p}\ B @ n' \in (\Gamma, \bar{p}\ A @ n) \qquad G;\Gamma, \bar{p}\ A @ n \vdash p\ B @ n'}{G;\Gamma \vdash p\ A @ n}\ \textsc{axCutBar}$$

4.2.1. *A L to DIL Translation.* In this section we show that every derivable L-sequent can be translated into a derivable DIL-sequent. Before giving the translation we will first show several admissibility results for DIL of inference rules which are similar to the ones we mentioned in Section 4. These two rules are required for the crucial left-to-right lemma. This lemma depends on the following admissible rule:

**Lemma 13** (Weakening)**.** If $G;\Gamma \vdash p_2\ B @ n$ is derivable, then $G;\Gamma, p_1\ A @ n_1 \vdash p_2\ B @ n_1$ is derivable.

*Proof.* This holds by straightforward induction on the assumed typing derivation. □

Note that we will use admissible rules as if they are inference rules of the logic throughout the sequel.

**Lemma 14** (Left-to-Right)**.** If $G;\Gamma_1, \bar{p}\ A @ n, \Gamma_2 \vdash \bar{p'}\ B @ n'$ is derivable, then so is $G;\Gamma_1, \Gamma_2, p'\ B @ n' \vdash p\ A @ n$.

*Proof.* Suppose $G;\Gamma_1, \bar{p}\ A @ n, \Gamma_2 \vdash \bar{p'}\ B @ n'$ is derivable and $\Gamma_3 =^{\text{def}} \Gamma_1, \bar{p}\ A @ n, \Gamma_2$. Then we derive $G;\Gamma_1, \Gamma_2, p'\ B @ n' \vdash p\ A @ n$ as follows:

$$\frac{p'\ B @ n' \in (\Gamma_3, p'\ B @ n') \qquad \dfrac{G;\Gamma_3 \vdash \bar{p'}\ B @ n'}{G;\Gamma_3, p'\ B @ n' \vdash \bar{p'}\ B @ n'}\ \textsc{Weakening}}{G;\Gamma_1, \Gamma_2, p'\ B @ n' \vdash p\ A @ n}\ \textsc{axCut}$$

[3]We will call a sequent in L a L-sequent and a sequent in DIL a DIL-sequent.

Thus, we obtain our result. □

We mentioned in the introduction that DIL avoids having to have rules like the monotonicity rules and other similar rules from L. To be able to translate every derivable sequent of L to DIL, we must show admissibility of those rules in DIL. The first of these admissible rules are the rules for reflexivity and transitivity.

**Lemma 15** (Reflexivity)**.** If $G, m \preccurlyeq_{p'} m; \Gamma \vdash p\ A @ n$ is derivable, then so is $G; \Gamma \vdash p\ A @ n$.

*Proof.* This holds by a straightforward induction on the form of the assumed derivation. □

**Lemma 16** (Transitivity)**.** If $G, n_1 \preccurlyeq_{p'} n_3; \Gamma \vdash p\ A @ n$ is derivable, $n_1 \preccurlyeq_{p'} n_2 \in G$ and $n_2 \preccurlyeq_{p'} n_3 \in G$, then $G; \Gamma \vdash p\ A @ n$ is derivable.

*Proof.* This holds by a straightforward induction on the form of the assumed derivation. □

There is not a trivial correspondence between conjunction in DIL and conjunction in L, because of the use of polarities in DIL. Hence, we must show that L's left rule for conjunction is indeed admissible in DIL.

**Lemma 17** (AndL)**.** If $G; \Gamma, \bar{p}\ A @ n \vdash p\ B @ n$ is derivable, then $G; \Gamma \vdash p\ (A \wedge_{\bar{p}} B) @ n$ is derivable.

*Proof.* This proof holds by directly deriving $G; \Gamma \vdash p\ (A \wedge_{\bar{p}} B) @ n$ in DIL. For the complete proof see Appendix A.1. □

L has several structural rules. The following lemmata show that all of these are admissible in DIL.

**Lemma 18** (Exchange)**.** If $G; \Gamma \vdash p\ A @ n$ is derivable and $\pi$ is a permutation of $\Gamma$, then $G; \pi\, \Gamma \vdash p\ A @ n$ is derivable.

*Proof.* This holds by a straightforward induction on the form of the assumed derivation. □

Note that we often leave the application of exchange implicit for readability.

**Lemma 19** (Contraction)**.** If $G; \Gamma, p\ A @ n, p\ A @ n, \Gamma' \vdash p'\ B @ n'$, then $G; \Gamma, p\ A @ n, \Gamma' \vdash p'\ B @ n'$.

*Proof.* This holds by a straightforward induction on the form of the assumed derivation. □

Monotonicity is taken as a primitive in L, but we have decided to leave monotonicity as an admissible rule in DIL. To show that it is admissible in DIL we need to be able to move nodes forward in the abstract Kripke graph. This is necessary to be able to satisfy the graph constraints in the rules IMP and IMPBAR when proving general monotonicity (Lemma 25). The next result is just weakening for the reachability judgment.

**Lemma 20** (Graph Weakening)**.** If $G \vdash n_1 \preccurlyeq^*_p n_2$, then $G, n_3 \preccurlyeq_{p'} n_4 \vdash n_1 \preccurlyeq^*_p n_2$.

*Proof.* This holds by a straightforward induction on the form of the assumed derivation. □

The function $\mathsf{raise}$ is an operation on abstract Kripke graphs that takes in two nodes $n_1$ and $n_2$, where $n_2$ is reachable from $n_1$, and then moves all the edges in an abstract Kripke graph forward to $n_2$. This essentially performs monotonicity on the given edges. It will be used to show that nodes in the context of a DIL-sequent can be moved forward using monotonicity resulting in a lemma called raising the lower bound logically (Lemma 24).

**Definition 21.** We define the function $\mathsf{raise}$ on abstract graphs as follows:

$$\begin{array}{lcl}
\mathsf{raise}\,(n_1, n_2, \cdot) & = & \cdot \\
\mathsf{raise}\,(n_1, n_2, (n_1 \preccurlyeq_p m, G)) & = & n_2 \preccurlyeq_p m, \mathsf{raise}\,(n_1, n_2, G) \\
\mathsf{raise}\,(n_1, n_2, (m \preccurlyeq_{\bar{p}} n_1, G)) & = & m \preccurlyeq_{\bar{p}} n_2, \mathsf{raise}\,(n_1, n_2, G) \\
\mathsf{raise}\,(n_1, n_2, (m \preccurlyeq_p m', G)) & = & m \preccurlyeq_p m', \mathsf{raise}\,(n_1, n_2, G), \text{where } m \not\equiv n_1 \text{ and } m' \not\equiv n_1. \\
\mathsf{raise}\,(n_1, n_2, (m \preccurlyeq_{\bar{p}} m', G)) & = & m \preccurlyeq_{\bar{p}} m', \mathsf{raise}\,(n_1, n_2, G), \text{where } m \not\equiv n_1 \text{ and } m' \not\equiv n_1.
\end{array}$$

**Lemma 22** (Raising the Lower Bound)**.** If $G \vdash n_1 \preccurlyeq_p^* n_2$ and $G, G_1 \vdash m \preccurlyeq_{p'}^* m'$, then $G, \mathsf{raise}\,(n_1, n_2, G_1) \vdash m \preccurlyeq_{p'}^* m'$.

*Proof.* This proof holds by induction on the form of $G, G_1 \vdash m \preccurlyeq_{p'}^* m'$. For the full proof see Appendix A.2. □

**Lemma 23** (Graph Node Containment)**.** If $G \vdash n_1 \preccurlyeq_p^* n_2$ and $n_1$ and $n_2$ are unique, then $n_1, n_2 \in |G|$.

*Proof.* This holds by straightforward induction on the form of $G \vdash n_1 \preccurlyeq_p^* n_2$. □

Finally, we arrive to raising the lower bound logically and general monotonicity. The latter depending on the former. These are the last of the admissibility results before showing that all translations of derivable L-sequents are derivable in DIL.

**Lemma 24** (Raising the Lower Bound Logically)**.** If $G, G_1, G'; \Gamma \vdash p\ A \ @\ n$ and $G, G' \vdash n_1 \preccurlyeq_p^* n_2$, then $G, \mathsf{raise}\,(n_1, n_2, G_1), G'; \Gamma \vdash p\ A \ @\ n$.

*Proof.* This proof holds by induction on the form of $G, G_1, G'; \Gamma \vdash p\ A \ @\ n$. For the full proof see Appendix A.3. □

**Lemma 25** (General Monotonicity)**.** If $G \vdash n_1 \preccurlyeq_{p_1}^* n_1', \ldots, G \vdash n_i \preccurlyeq_{p_i}^* n_i', G \vdash m \preccurlyeq_p^* m'$, and $G; \bar{p}_1\ A_1 \ @\ n_1, \ldots, \bar{p}_i\ A_i \ @\ n_i \vdash p\ B \ @\ m$, then $G; \bar{p}_1 A_1 @ n_1', \ldots, \bar{p}_i A_i @ n_i' \vdash p\ B \ @\ m'$.

*Proof.* This proof holds by induction on the form of $G; \bar{p}_1\ A_1 \ @\ n_1, \ldots, \bar{p}_i\ A_i \ @\ n_i \vdash p\ B \ @\ m$. For the full proof see Appendix A.4. □

The following are corollaries of general monotonicity. The latter two corollaries show that the monotonicity rules of L are admissible in DIL.

**Corollary 26** (Monotonicity)**.** Suppose $G \vdash n_1 \preccurlyeq_p^* n_2$. Then

i. if $G; \Gamma, \bar{p}\ A \ @\ n_1, \Gamma' \vdash p'\ B \ @\ n'$, then $G; \Gamma, \bar{p}\ A \ @\ n_2, \Gamma' \vdash p'\ B \ @\ n'$, and
ii. if $G; \Gamma \vdash p\ A \ @\ n_1$, then $G; \Gamma \vdash p\ A \ @\ n_2$.

**Corollary 27** (MonoL)**.** If $G; \Gamma, p\ A \ @\ n_1, p\ A \ @\ n_2, \Gamma' \vdash p'\ B \ @\ n'$ is derivable and $n_1 \preccurlyeq_p n_2 \in G$, then $G; \Gamma, p\ A \ @\ n_1, \Gamma' \vdash p'\ B \ @\ n'$ is derivable.

*Proof.* This result easily follows by part one of Corollary 26, and contraction (Lemma 19). □

**Corollary 28** (MonoR)**.** If $G; \Gamma, \bar{p}\ A \mathbin{@} n_1, \Gamma' \vdash p\ A \mathbin{@} n_2$ and $n_1 \preccurlyeq_p n_2 \in G$, then $G; \Gamma, \Gamma' \vdash p\ A \mathbin{@} n_2$ is derivable.

*Proof.* Suppose $G; \Gamma, \bar{p}\ A \mathbin{@} n_1, \Gamma' \vdash p\ A \mathbin{@} n_2$ and $n_1 \preccurlyeq_p n_2 \in G$. Then by part one of monotonicity (Corollary 26) we know $G; \Gamma, \bar{p}\ A \mathbin{@} n_2, \Gamma' \vdash p\ A \mathbin{@} n_2$. Finally, we know by the axiom cut rule that $G; \Gamma, \Gamma' \vdash p\ A \mathbin{@} n_2$. □

We now have everything we need to prove that every derivable sequent of L can be translated to a derivable sequent in DIL. The following definition defines the translation from L into DIL.

**Definition 29.** The following defines a translation of formulas of L to formulas of DIL:

$$\begin{array}{rclrclrcl}
\mathsf{D}(\top) & = & \langle + \rangle & \mathsf{D}(A \land B) & = & \mathsf{D}(A) \land_+ \mathsf{D}(B) & \mathsf{D}(A \supset B) & = & \mathsf{D}(A) \to_+ \mathsf{D}(B) \\
\mathsf{D}(\perp) & = & \langle - \rangle & \mathsf{D}(A \lor B) & = & \mathsf{D}(A) \land_- \mathsf{D}(B) & \mathsf{D}(B \prec A) & = & \mathsf{D}(A) \to_- \mathsf{D}(B)
\end{array}$$

Next we extend the previous definition to contexts:

$$\begin{array}{lcl}
\mathsf{D}(\cdot)^p & = & \cdot \\
\mathsf{D}(n : A, \Gamma)^p & = & p\, \mathsf{D}(A) \mathbin{@} n, \mathsf{D}(\Gamma)^p
\end{array}$$

The following defines the translation of graphs:

$$\begin{array}{lcl}
\mathsf{D}(\cdot) & = & \cdot \\
\mathsf{D}((n_1, n_2), G) & = & n_1 \preccurlyeq_+ n_2, \mathsf{D}(G)
\end{array}$$

The translation of a L-sequent is a DIL-sequent which requires a particular formula as the active formula. The following defines such a translation:

An activation of a L-sequent $\Gamma \vdash_G \Delta$ is a DIL-sequent
$\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta_1, \Delta_2)^- \vdash + \mathsf{D}(A) \mathbin{@} n$, where $\Delta = \Delta_1, n : A, \Delta_2$.

The previous definition implies the following result:

**Lemma 30** (Reachability)**.** If $n_1 G n_2$, then $\mathsf{D}(G) \vdash n_1 \preccurlyeq_+^* n_2$.

The following result shows that every derivable L-sequent can be translated into a derivable DIL-sequent. We do this by considering an arbitrary activation of the L-sequent, and then show that this arbitrary activation is derivable in DIL, but if it so happens that this is not the correct activation, then we can always get the correct one by using the left-to-right lemma (Lemma 14) to switch out the active formula.

**Lemma 31** (Containment of L in DIL)**.** If $\mathsf{D}(G); \Gamma' \vdash + A \mathbin{@} n$ is an activation of the derivable L-sequent $\Gamma \vdash_G \Delta$, then $\mathsf{D}(G); \Gamma' \vdash + A \mathbin{@} n$ is derivable.

*Proof.* This proof holds by induction on the form of the sequent $\Gamma \vdash_G \Delta$. For the full proof see Appendix A.5. □

4.2.2. *A DIL to L Translation.* This section is similar to the previous one, but we give a translation of DIL-sequents to L-sequents. We first have the definition of the translation from DIL to L.

**Definition 32.** The following defines a translation of formulas of DIL to formulas of L:

$$\begin{array}{rclcrclcrcl}
\mathsf{L}(\langle + \rangle) & = & \top & \quad & \mathsf{L}(A \wedge_+ B) & = & \mathsf{L}(A) \wedge \mathsf{L}(B) & \quad & \mathsf{L}(A \rightarrow_+ B) & = & \mathsf{L}(A) \supset \mathsf{L}(B) \\
\mathsf{L}(\langle - \rangle) & = & \bot & & \mathsf{L}(A \wedge_- B) & = & \mathsf{L}(A) \vee \mathsf{L}(B) & & \mathsf{L}(B \rightarrow_- A) & = & \mathsf{L}(A) \prec \mathsf{L}(B)
\end{array}$$

Next we extend the previous definition to positive and negative contexts:

$$\begin{array}{rcl}
\mathsf{L}(+\,A \mathbin{@} n, \Gamma)^+ & = & n : \mathsf{L}(A), \mathsf{L}(\Gamma)^+ \\
\mathsf{L}(-\,A \mathbin{@} n, \Gamma)^+ & = & \mathsf{L}(\Gamma)^+ \\
\\
\mathsf{L}(-\,A \mathbin{@} n, \Gamma)^- & = & n : \mathsf{L}(A), \mathsf{L}(\Gamma)^- \\
\mathsf{L}(+\,A \mathbin{@} n, \Gamma)^- & = & \mathsf{L}(\Gamma)^-
\end{array}$$

The following defines the translation of graphs:

$$\begin{array}{rcl}
\mathsf{L}(n_1 \preccurlyeq_+ n_2, G) & = & (n_1, n_2), \mathsf{L}(G) \\
\mathsf{L}(n_2 \preccurlyeq_- n_1, G) & = & (n_1, n_2), \mathsf{L}(G)
\end{array}$$

Finally, the following defines the translation of DIL sequents:

$$\begin{array}{rcl}
\mathsf{L}(G; \Gamma \vdash +\,A \mathbin{@} n) & = & \mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n : A, \mathsf{L}(\Gamma)^- \\
\mathsf{L}(G; \Gamma \vdash -\,A \mathbin{@} n) & = & \mathsf{L}(\Gamma)^+, n : A \vdash_{\mathsf{L}(G)} \mathsf{L}(\Gamma)^-
\end{array}$$

Next we have a few admissible rules that are needed to complete the proof of containment of DIL in L.

**Lemma 33** (Left and Right Weakening in L)**.**

WEAKL: If $\Gamma, n : A \vdash_G \Delta$, then $\Gamma, n : A, n : B \vdash_G \Delta$.

WEAKR: If $\Gamma \vdash_G n : A, \Delta$, then $\Gamma \vdash_G n : A, n : B, \Delta$.

*Proof.* Both parts of this result hold by straightforward induction on the assumed derivation. □

**Lemma 34** (Left and Right Contraction in L)**.**

CONTRL: If $\Gamma_1, n : A, \Gamma_2, n : A, \Gamma_3 \vdash_G \Delta$, then $\Gamma_1, n : A, \Gamma_2, \Gamma_3 \vdash_G \Delta$.

CONTRR: If $\Gamma \vdash_G \Delta_1, n : A, \Delta_2, n : A, \Delta_3$, then $\Gamma \vdash_G \Delta_1, \Delta_2, n : A, \Delta_3$.

*Proof.* Both parts of this result hold by straightforward induction on the assumed derivation. □

**Lemma 35** (Reachability Weakening in DIL)**.** For any $n_1, n_2 \in |n \preccurlyeq_p n, G|, |\Gamma|$ if $G \vdash n_1 \preccurlyeq_p^* n_2$ and $G; \Gamma \vdash p\,A \mathbin{@} n$, then $G, n_1 \preccurlyeq_p n_2; \Gamma \vdash p\,A \mathbin{@} n$.

*Proof.* By straightforward induction on the form of $G; \Gamma \vdash p\,A \mathbin{@} n$. □

Finally, the next two results show that every derivable DIL-sequent can be translated into a derivable L-sequent. One interesting aspect of these results is that DIL inference rules where the active formula is positive correspond to the right-inference rules of L, and when the active formula is negative correspond to left-inference rules of L. In addition, the use of axiom cuts in DIL correspond to uses of contraction in L.

**Lemma 36.** Suppose $G;\Gamma \vdash p\ A \mathbin{@} n$ is a derivable DIL-sequent such that for any $n_1, n_2 \in |n \preccurlyeq_{p'} n, G|, |\Gamma|$ if $G \vdash n_1 \preccurlyeq^*_{p'} n_2$, then $n_1 \preccurlyeq_{p'} n_2 \in G$. Then by using the definition of the translation of DIL-sequents we have that $\mathsf{L}(G;\Gamma \vdash p\ A \mathbin{@} n)$ is a derivable L-sequent.

*Proof.* This proof holds by induction on the assumed derivation. For the full proof see Appendix A.6. □

**Lemma 37** (Containment of DIL in L)**.** Suppose $G;\Gamma \vdash p\ A \mathbin{@} n$ is a derivable DIL-sequent. Then there exists an abstract Kripke graph $G'$, such that, $\mathsf{L}(\mathsf{G}';\Gamma \vdash p\ A \mathbin{@} n)$.

*Proof.* Suppose $G;\Gamma \vdash p\ A \mathbin{@} n$ is a derivable DIL-sequent. Then by repeatedly applying Reachability Weakening in DIL (Lemma 35), which can only be applied a finite number of times before reaching a fixed point, we will obtain a derivation $G'';\Gamma \vdash p\ A \mathbin{@} n$ satisfying the condition:

$$\text{for any } n_1, n_2 \in |n \preccurlyeq_{p'} n, G''|, |\Gamma|, \text{ if } G'' \vdash n_1 \preccurlyeq^*_{p'} n_2, \text{ then } n_1 \preccurlyeq_{p'} n_2 \in G''$$

Choose $G' = G''$. Then we obtain our result by applying Lemma 36 to $G';\Gamma \vdash p\ A \mathbin{@} n$. □

4.2.3. *Completeness.* We now use the previous translations as a means to exploit the completeness result of L. The following definition and lemma relate the two translations which will be needed by our main results of this section.

**Definition 38.** We say two abstract Kripke graphs, $G_1$ and $G_2$, are **isomorphic** iff for any $n_1 \preccurlyeq_p n_2 \in G_1$, $n_1 \preccurlyeq_p n_2 \in G_2$ or $n_2 \preccurlyeq_{\bar{p}} n_1 \in G_2$, and for any $n_1 \preccurlyeq_p n_2 \in G_2$, $n_1 \preccurlyeq_p n_2 \in G_1$ or $n_2 \preccurlyeq_{\bar{p}} n_1 \in G_1$.

**Lemma 39** (L and D Relationships)**.**
i. For any abstract Kripke graph $G$, $\mathsf{D}(\mathsf{L}(G))$ is isomorphic to $G$.
ii. For any abstract Kripke graph, $\mathsf{L}(\mathsf{D}(G)) = G$.
iii. For any DIL-formula $A$, $\mathsf{D}(\mathsf{L}(A)) = A$.
iv. For any L-formula $A$, $\mathsf{L}(\mathsf{D}(A)) = A$.

*Proof.* Part i and ii follow directly by induction on $G$, and part iii and iv follow directly by induction on $A$. □

It is straightforward to extend the previous result to contexts in both DIL and L.

The interpretation of L-formulas into a Kripke model is identical to the interpretation of DIL-formulas. Thus, we use the same syntax to denote the interpretation of an L-formula. In fact, we have the following straightforward result.

**Lemma 40.** Suppose $(W, R, V)$ is a Kripke model and $N$ is a node interpreter. Then the following hold:
i. $[\![A]\!]_{(N\,n)}$ iff $[\![\mathsf{L}(A)]\!]_{N\,n}$.
ii. $[\![G]\!]_N$ iff for any $n_1 \mathsf{L}(G) n_2$, $R\,(N\,n_1)\,(N\,n_2)$.
iii. $[\![\Gamma]\!]_N$ iff for any $n : \mathsf{L}(A) \in \mathsf{L}(\Gamma)^+$, $[\![\mathsf{L}(A)]\!]_{N\,n}$, and for any $n : \mathsf{L}(A) \in \mathsf{L}(\Gamma)^-$, $\neg[\![\mathsf{L}(A)]\!]_{N\,n}$.

$$
\begin{array}{lrcl}
\text{(indices)} & d & ::= & 1 \,|\, 2 \\
\text{(polarities)} & p & ::= & + \,|\, - \\
\text{(types)} & A, B, C & ::= & \langle p \rangle \,|\, A \to_p B \,|\, A \land_p B \\
\text{(terms)} & t & ::= & x \,|\, \mathbf{triv} \,|\, (t, t') \,|\, \mathbf{in}_d\, t \,|\, \lambda x.t \,|\, \langle t, t' \rangle \,|\, \nu\, x.t \bullet t' \\
\text{(canonical terms)} & c & ::= & x \mid \mathbf{triv} \mid (t, t') \mid \mathbf{in}_d\, t \mid \lambda x.t \mid \langle t, t' \rangle \\
\text{(graphs)} & G & ::= & \cdot \,|\, n \preccurlyeq_p n' \,|\, G, G' \\
\text{(contexts)} & \Gamma & ::= & \cdot \,|\, x : p\, A \mathbin{@} n \,|\, \Gamma, \Gamma'
\end{array}
$$

Figure 6: Syntax for DTT.

We recall the definition of validity in L due to Pinto and Uustalu [28].

**Definition 41** (Counter Models and L-validity (p. 6, Definition 1, [28]))**.** A Kripke model $(W, R, V)$ and node interpreter $N$ is a **counter-model** to a L-sequent $\Gamma \vdash_G \Delta$, if

i. for any $n_1 G n_2$, $R\,(N\,n_1)\,(N\,n_2)$;
ii. for any $n : A \in \Gamma$, $[\![A]\!]_{N\,n}$; and
iii. for any $n : B \in \Delta$, $\neg[\![B]\!]_{N\,n}$.

The L-sequent is **L-valid** if it has no counter-models.

The following lemma relates validity of DIL to validity of L, and is the key to proving completeness of DIL.

**Lemma 42** (DIL-validity is L-validity)**.** Suppose $[\![G; \Gamma \vdash p\,A \mathbin{@} n]\!]_N$ holds for some Kripke model $(W, R, V)$ and node interpreter $N$ on $|G|$. Then by using the translation of DIL-sequents from Definition 4.2.2 we have that $\mathsf{L}(G; \Gamma \vdash p\,A \mathbin{@} n)$ is L-valid.

*Proof.* This result holds essentially by definition. For the full proof see Appendix A.7. □

Finally, we have completeness of DIL by connecting all of the results of this section.

**Theorem 43** (Completeness)**.** Suppose $(W, R, V)$ is a Kripke model and $N$ is a node interpreter. If $[\![G; \Gamma \vdash p\,A \mathbin{@} n]\!]_N$ holds, then $G; \Gamma \vdash p\,A \mathbin{@} n$ is derivable.

*Proof.* Suppose $(W, R, V)$ is a Kripke model and $N$ is a node interpreter. Furthermore, suppose $[\![G; \Gamma \vdash p\,A \mathbin{@} n]\!]_N$ holds. Let $p = +$. By Lemma 42 we know $\mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n : \mathsf{L}(A), \mathsf{L}(\Gamma)^-$ is valid, and by completeness of L (Corollary 1, p. 13, [28]) we know $\mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n : \mathsf{L}(A), \mathsf{L}(\Gamma)^-$ is derivable. By containment of L in DIL (Lemma 31) we know that the activation $\mathsf{D}(\mathsf{L}(G)); \mathsf{D}(\mathsf{L}(\Gamma)^+)^+, \mathsf{D}(\mathsf{L}(\Gamma)^-)^- \vdash +\,\mathsf{D}(\mathsf{L}(A)) \mathbin{@} n$ is derivable. Finally, by Lemma 39 we can see that the former sequent is equivalent to $G; \Gamma \vdash p\,A \mathbin{@} n$, and thus, we obtain our result. The case when $p = -$ is similar, but before using Lemma 39 one must first use the left-to-right admissible rule (Lemma 14). □

## 5. Dualized Type Theory (DTT)

In this section we give DIL a term assignment yielding Dualized Type Theory (DTT). First, we introduce DTT, and give several examples illustrating how to program in DTT. Then we present the metatheory of DTT.

The syntax for DTT is defined in Figure 6. Polarities, types, and graphs are all the same as they were in DIL. Contexts differ only by the addition of labeling each hypothesis with a variable. Terms, denoted $t$, consist of introduction forms, together with cut terms

$$\frac{G \vdash n \preccurlyeq_p^* n'}{G; \Gamma, x : p\, A \,@\, n, \Gamma' \vdash x : p\, A \,@\, n'}\ \textsc{Ax} \qquad \frac{}{G; \Gamma \vdash \mathbf{triv} : p\, \langle p \rangle \,@\, n}\ \textsc{Unit}$$

$$\frac{G; \Gamma \vdash t_1 : p\, A \,@\, n \qquad G; \Gamma \vdash t_2 : p\, B \,@\, n}{G; \Gamma \vdash (t_1, t_2) : p\, (A \wedge_p B) \,@\, n}\ \textsc{And}$$

$$\frac{G; \Gamma \vdash t : p\, A_d \,@\, n}{G; \Gamma \vdash \mathbf{in}_d\, t : p\, (A_1 \wedge_{\bar{p}} A_2) \,@\, n}\ \textsc{AndBar}$$

$$\frac{\begin{array}{l} n' \notin |G|, |\Gamma| \\ (G, n \preccurlyeq_p n'); \Gamma, x : p\, A \,@\, n' \vdash t : p\, B \,@\, n' \end{array}}{G; \Gamma \vdash \lambda x.t : p\, (A \to_p B) \,@\, n}\ \textsc{Imp}$$

$$\frac{\begin{array}{l} G \vdash n \preccurlyeq_{\bar{p}}^* n' \\ G; \Gamma \vdash t_1 : \bar{p}\, A \,@\, n' \qquad G; \Gamma \vdash t_2 : p\, B \,@\, n' \end{array}}{G; \Gamma \vdash \langle t_1, t_2 \rangle : p\, (A \to_{\bar{p}} B) \,@\, n}\ \textsc{ImpBar}$$

$$\frac{\begin{array}{c} G; \Gamma, x : \bar{p}\, A \,@\, n \vdash t_1 : +\, B \,@\, n' \\ G; \Gamma, x : \bar{p}\, A \,@\, n \vdash t_2 : -\, B \,@\, n' \end{array}}{G; \Gamma \vdash \nu\, x.t_1 \bullet t_2 : p\, A \,@\, n}\ \textsc{Cut}$$

Figure 7: Type-Assignment Rules for DTT.

$\nu\, x.t \bullet t'$[4]. We denote variables as $x$, $y$, $z$, .... The term **triv** is the introduction form for units, $(t, t')$ is the introduction form for pairs, similarly the terms $\mathbf{in}_1\, t$ and $\mathbf{in}_2\, t$ introduce disjunctions, $\lambda x.t$ introduces implication, and $\langle t, t' \rangle$ introduces co-implication. The type-assignment rules are defined in Figure 7, and result from a simple term assignment to the rules for DIL. Finally, the reduction rules for DTT are defined in Figure 8. The reduction rules should be considered rewrite rules that can be applied anywhere within a term. (The congruence rules are omitted.)

Programming in DTT is not functional programming as usual, so we now give several illustrative examples. The reader familiar with type theories based on sequent calculi will find the following very familiar. The encodings are similar to that of Curien and Herbelin's $\bar{\lambda}\mu\tilde{\mu}$-calculus [9]. The locus of computation is the cut term, so naturally, function application is modeled using cuts. Suppose

$$\begin{array}{lll} D_1 & =^{\text{def}} & G; \Gamma \vdash \lambda x.t : +\, (A \to_+ B) \,@\, n \\ D_2 & =^{\text{def}} & G; \Gamma \vdash t' : +\, A \,@\, n \\ \Gamma' & =^{\text{def}} & \Gamma, y : -\, B \,@\, n \end{array}$$

Then we can construct the following typing derivation:

[4] In classical type theories the symbol $\mu$ usually denotes cut, but we have reserved that symbol – indexed by a polarity – to be used with inductive (positive polarity) and coinductive (negative polarity) types in future work.

$$\frac{}{\nu\, z.\lambda x.t \bullet \langle t_1, t_2\rangle \rightsquigarrow \nu\, z.[t_1/x]t \bullet t_2}\ \textsc{RImp}$$

$$\frac{}{\nu\, z.\langle t_1, t_2\rangle \bullet \lambda x.t \rightsquigarrow \nu\, z.t_2 \bullet [t_1/x]t}\ \textsc{RImpBar} \qquad \frac{}{\nu\, z.(t_1, t_2) \bullet \mathbf{in}_1\, t \rightsquigarrow \nu\, z.t_1 \bullet t}\ \textsc{RAnd1}$$

$$\frac{}{\nu\, z.(t_1, t_2) \bullet \mathbf{in}_2\, t \rightsquigarrow \nu\, z.t_2 \bullet t}\ \textsc{RAnd2} \qquad \frac{}{\nu\, z.\mathbf{in}_1\, t \bullet (t_1, t_2) \rightsquigarrow \nu\, z.t \bullet t_1}\ \textsc{RAndBar1}$$

$$\frac{}{\nu\, z.\mathbf{in}_2\, t \bullet (t_1, t_2) \rightsquigarrow \nu\, z.t \bullet t_2}\ \textsc{RAndBar2} \qquad \frac{x \notin \mathsf{FV}\,(t)}{\nu\, x.t \bullet x \rightsquigarrow t}\ \textsc{RRet}$$

$$\frac{}{\nu\, z.(\nu\, x.t_1 \bullet t_2) \bullet t \rightsquigarrow \nu\, z.[t/x]t_1 \bullet [t/x]t_2}\ \textsc{RBetaL}$$

$$\frac{}{\nu\, z.c \bullet (\nu\, x.t_1 \bullet t_2) \rightsquigarrow \nu\, z.[c/x]t_1 \bullet [c/x]t_2}\ \textsc{RBetaR}$$

Figure 8: Reduction Rules for DTT.

$$\dfrac{D_1 \qquad \dfrac{D_2 \qquad \dfrac{}{G; \Gamma' \vdash y : -\, B \,@\, n}\ \textsc{ax}}{G; \Gamma' \vdash \langle t', y\rangle : -\,(A \rightarrow_+ B) \,@\, n}\ \textsc{impBar}}{G; \Gamma \vdash \nu\, y.\lambda x.t \bullet \langle t', y\rangle : +\, B \,@\, n}\ \textsc{cut}$$

Implication was indeed eliminated, yielding the conclusion.

There is some intuition one can use while thinking about this style of programming that is based on the encoding of classical logic – Parigot's $\lambda\mu$-calculus – into the $\pi$-calculus. See for example [35, 17]. We can think of positive variables as input ports, and negative variables as output ports. Clearly, these notions are dual. Then a cut of the form $\nu\, z.t \bullet t'$ can be intuitively understood as a device capable of routing information. We think of this term as first running the term $t$, and then plugging its value into the continuation $t'$. Thus, negative terms are continuations. Now consider the instance of the previous term $\nu\, z.t \bullet y$ where $t$ is a positive term and $y$ is a negative variable (an output port). This can be intuitively understood as after running $t$, route its value through the output port $y$. Now consider the instance $\nu\, z.t \bullet z$. This term can be understood as after running the term $t$, route its value through the output port $z$, but then capture this value as the return value. Thus, the cut term reroutes output ports into the actual return value of the cut.

There is one additional bit of intuition we can use when thinking about programming in DTT. We can think of cuts of the form $\nu z.(\lambda x_1 \cdots \lambda x_i.t) \bullet \langle t_1, \langle t_2, \cdots \langle t_i, z\rangle \cdots \rangle$ as an abstract machine, where $\lambda x_1 \cdots \lambda x_i.t$ is the functional part of the machine, and $\langle t_1, \langle t_2, \cdots \langle t_i, z\rangle \cdots \rangle$ is the stack of inputs the abstract machine will apply the function to ultimately routing the final result of the application through $z$, but rerouting this into the return value. This intuition is not new, but was first observed by Curien and Herbelin in [9]; see also [10].

Similarly to the eliminator for implication we can define the eliminator for disjunction in the form of the usual case analysis. Suppose $G; \Gamma \vdash t : +\,(A \wedge_- B) \,@\, n$, $G; \Gamma, x : +\, A \,@\, n \vdash t_1 : +\, C \,@\, n$, and $G; \Gamma, x : +\, B \,@\, n \vdash t_2 : +\, C \,@\, n$ are all admissible. Then we can derive the usual eliminator for disjunction. Define $\mathbf{case}\, t\, \mathbf{of}\, x.t_1, x.t_2 =^{\text{def}} \nu\, z_0.(\nu\, z_1.(\nu\, z_2.t \bullet (z_1, z_2)) \bullet (\nu\, x.t_2 \bullet z_0)) \bullet (\nu\, x.t_1 \bullet z_0)$. Then we have the following result.

**Lemma 44.** The following rule is derivable:

$$\frac{\begin{array}{l} G;\Gamma, x : p\, A \mathbin{@} n \vdash t_1 : p\, C \mathbin{@} n \\ G;\Gamma, x : p\, B \mathbin{@} n \vdash t_2 : p\, C \mathbin{@} n \qquad G;\Gamma \vdash t : p\,(A \wedge_{\bar{p}} B) \mathbin{@} n \end{array}}{G;\Gamma \vdash \mathbf{case}\, t\, \mathbf{of}\, x.t_1, x.t_2 : p\, C \mathbin{@} n}\ \textsc{case}$$

*Proof.* A full derivation in DTT can be found in Appendix B.1. □

Now consider the term $\nu\, x.\mathbf{in}_1\,(\nu\, y.\mathbf{in}_2\, \langle y, \mathbf{triv}\rangle \bullet x) \bullet x$. This term is the inhabitant of the type $A \wedge_{-} \sim A$, and its typing derivation follows from the derivation given in Section 4. We can see by looking at the syntax that the cuts involved are indeed on the axiom $x$, thus this term has no canonical form. In [8] Crolard shows that inhabitants such as these amount to a constructive coroutine. That is, it is a restricted form of a continuation.

We now consider several example reductions in DTT. In the following examples we underline non-top-level redexes. The first example simply $\alpha$-converts the function $\lambda x.x$ into $\lambda z.z$ as follows:

$$\begin{array}{rcl} \lambda z.\underline{\nu\, y.\lambda x.x \bullet \langle z, y\rangle} & \overset{(\textsc{RImp})}{\rightsquigarrow} & \lambda z.\underline{\nu\, y.z \bullet y} \\ & \overset{(\textsc{RRet})}{\rightsquigarrow} & \lambda z.z \end{array}$$

A more involved example is the application of the function $\lambda x.(\lambda y.y)$ to the arguments **triv** and **triv**.

$$\begin{array}{rcl} \nu\, z.\lambda x.(\lambda y.y) \bullet \langle \mathbf{triv}, \langle \mathbf{triv}, z\rangle\rangle & \overset{(\textsc{RImp})}{\rightsquigarrow} & \nu\, z.\lambda y.y \bullet \langle \mathbf{triv}, z\rangle \\ & \overset{(\textsc{RImp})}{\rightsquigarrow} & \nu\, z.\mathbf{triv} \bullet z \\ & \overset{(\textsc{RRet})}{\rightsquigarrow} & \mathbf{triv} \end{array}$$

## 6. Metatheory of DTT

We now present the basic metatheory of DTT, starting with type preservation. We begin with the inversion lemma which is necessary for proving type preservation.

**Lemma 45** (Inversion)**.**
i. If $G;\Gamma \vdash (t_1, t_2) : p\,(A \wedge_p B) \mathbin{@} n$, then $G;\Gamma \vdash t_1 : p\, A \mathbin{@} n$ and $G;\Gamma \vdash t_2 : p\, B \mathbin{@} n$.
ii. If $G;\Gamma \vdash \mathbf{in}_d\, t : p\,(A_1 \wedge_{\bar{p}} A_2) \mathbin{@} n$, then $G;\Gamma \vdash t : p\, A_d \mathbin{@} n$.
iii. If $G;\Gamma \vdash \lambda x.t : p\,(A \rightarrow_p B) \mathbin{@} n$, then $(G, n \preccurlyeq_p n');\Gamma, x : p\, A \mathbin{@} n' \vdash t : p\, B \mathbin{@} n'$ for any $n' \notin |G|, |\Gamma|$.
iv. If $G;\Gamma \vdash \langle t_1, t_2\rangle : p\,(A \rightarrow_{\bar{p}} B) \mathbin{@} n$, then $G \vdash n \preccurlyeq^*_{\bar{p}} n'$, $G;\Gamma \vdash t_1 : \bar{p}\, A \mathbin{@} n'$, and $G;\Gamma \vdash t_2 : p\, B \mathbin{@} n'$ for some node $n'$.

*Proof.* Each case of the above lemma holds by a trivial proof by induction on the assumed typing derivation. □

The results node substitution and substitution for typing are essential for the cases of type preservation that reduce a top-level redex. Node substitution, denoted $[n_1/n_2]n$, is defined as follows:

$$\begin{array}{lcl} [n_1/n_2]n_2 & = & n_1 \\ [n_1/n_2]n & = & n \text{ where } n \text{ is distinct from } n_2 \end{array}$$

The following lemmas are necessary in the proof of node substitution for typing.

**Lemma 46** (Node Renaming)**.** If $G_1, G_2 \vdash n_1 \preccurlyeq^*_p n_3$, then for any nodes $n_4$ and $n_5$, we have $[n_4/n_5]G_1, [n_4/n_5]G_2 \vdash [n_4/n_5]n_1 \preccurlyeq^*_p [n_4/n_5]n_3$.

$$\frac{}{\Gamma, x : p\ A, \Gamma' \vdash_c x : p\ A}\ \textsc{ClassAx} \qquad \frac{}{\Gamma \vdash_c \mathbf{triv} : p\ \langle p \rangle}\ \textsc{ClassUnit}$$

$$\frac{\Gamma \vdash_c t_1 : p\ A \qquad \Gamma \vdash_c t_2 : p\ B}{\Gamma \vdash_c (t_1, t_2) : p\ (A \wedge_p B)}\ \textsc{ClassAnd}$$

$$\frac{\Gamma \vdash_c t : p\ A_d}{\Gamma \vdash_c \mathbf{in}_d\ t : p\ (A_1 \wedge_{\bar{p}} A_2)}\ \textsc{ClassAndBar} \qquad \frac{\Gamma, x : p\ A \vdash_c t : p\ B}{\Gamma \vdash_c \lambda x.t : p\ (A \to_p B)}\ \textsc{ClassImp}$$

$$\frac{\Gamma \vdash_c t_1 : \bar{p}\ A \qquad \Gamma \vdash_c t_2 : p\ B}{\Gamma \vdash_c \langle t_1, t_2 \rangle : p\ (A \to_{\bar{p}} B)}\ \textsc{ClassImpBar} \qquad \frac{\begin{array}{c}\Gamma, x : \bar{p}\ A \vdash_c t_1 : +\ B \\ \Gamma, x : \bar{p}\ A \vdash_c t_2 : -\ B\end{array}}{\Gamma \vdash_c \nu\, x.t_1 \bullet t_2 : p\ A}\ \textsc{ClassCut}$$

Figure 9: Classical typing of DTT terms

*Proof.* This proof holds by induction on the assumed reachability derivation. For the full proof see Appendix C.1. □

**Lemma 47** (Node Substitution for Reachability)**.** If $G, n_1 \preccurlyeq_{p_1} n_2, G' \vdash n_4 \preccurlyeq^*_p n_5$ and $G, G' \vdash n_1 \preccurlyeq^*_{p_1} n_3$, then $[n_3/n_2]G, [n_3/n_2]G' \vdash [n_3/n_2]n_4 \preccurlyeq^*_p [n_3/n_2]n_5$.

*Proof.* This proof holds by by induction on the form of the assumed reachability derivation. For the full proof see Appendix C.2. □

**Lemma 48** (Node Substitution for Typing)**.** If $G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma \vdash t : p_2\ A\ @\ n_3$ and $G, G' \vdash n_1 \preccurlyeq^*_{p_1} n_4$, then $[n_4/n_2]G, [n_4/n_2]G'; [n_4/n_2]\Gamma \vdash t : p_2\ A\ @\ [n_4/n_2]n_3$.

*Proof.* This holds by induction on the form of the assumed typing derivation. See Appendix C.3 for the full proof. □

The next lemma is crucial for type preservation.

**Lemma 49** (Substitution for Typing)**.** If $G; \Gamma \vdash t_1 : p_1\ A\ @\ n_1$ and $G; \Gamma, x : p_1\ A\ @\ n_1, \Gamma' \vdash t_2 : p_2\ B\ @\ n_2$, then $G; \Gamma, \Gamma' \vdash [t_1/x]t_2 : p_2\ B\ @\ n_2$.

*Proof.* This proof holds by induction on the second assumed typing relation. For the full proof see Appendix C.4. □

**Lemma 50** (Type Preservation)**.** If $G; \Gamma \vdash t : p\ A\ @\ n$, and $t \rightsquigarrow t'$, then $G; \Gamma \vdash t' : p\ A\ @\ n$.

*Proof.* This proof holds by induction on the form of the assumed typing derivation. For the full proof see Appendix C.5. □

A more substantial property is strong normalization of reduction for typed terms. To prove this result, we will prove a stronger property, namely strong normalization for reduction of terms which are typable using the system of classical typing rules in Figure 9 [7]. This is justified by the following easy result (proof omitted), where $\mathsf{DN}(\Gamma)$ just drops the world annotations from assumptions in $\Gamma$:

**Theorem 51.** If $G; \Gamma \vdash t : p\ A\ @\ n$, then $\mathsf{DN}(\Gamma) \vdash_c t : p\ A$

Let **SN** be the set of terms which are strongly normalizing with respect to the reduction relation. Let *Var* be the set of term variables, and let us use $x$ and $y$ as metavariables for

$$
\begin{array}{lll}
t \in [\![A]\!]^{+} & \Leftrightarrow & \forall x \in \mathit{Var}.\ \forall t' \in [\![A]\!]^{-}.\ \nu\, x.t \bullet t' \in \mathbf{SN} \\
t \in [\![A]\!]^{-} & \Leftrightarrow & \forall x \in \mathit{Var}.\ \forall t' \in [\![A]\!]^{+c}.\ \nu\, x.t' \bullet t \in \mathbf{SN} \\
t \in [\![\langle + \rangle]\!]^{+c} & \Leftrightarrow & t \in \mathit{Var}\ \vee\ t \equiv \mathbf{triv} \\
t \in [\![\langle - \rangle]\!]^{+c} & \Leftrightarrow & t \in \mathit{Var} \\
t \in [\![A \rightarrow_{+} B]\!]^{+c} & \Leftrightarrow & t \in \mathit{Var}\ \vee\ \exists x, t'.t \equiv \lambda x.t'\ \wedge\ \forall t'' \in [\![A]\!]^{+}.\ [t''/x]t' \in [\![B]\!]^{+} \\
t \in [\![A \rightarrow_{-} B]\!]^{+c} & \Leftrightarrow & t \in \mathit{Var}\ \vee\ \exists t_1 \in [\![A]\!]^{-}, t_2 \in [\![B]\!]^{+}.\ t \equiv \langle t_1, t_2 \rangle \\
t \in [\![A \wedge_{+} B]\!]^{+c} & \Leftrightarrow & t \in \mathit{Var}\ \vee\ \exists t_1 \in [\![A]\!]^{+}, t_2 \in [\![B]\!]^{+}.\ t \equiv (t_1, t_2) \\
t \in [\![A_1 \wedge_{-} A_2]\!]^{+c} & \Leftrightarrow & t \in \mathit{Var}\ \vee\ \exists d.\exists t' \in [\![A_d]\!]^{+}.\ t \equiv \mathbf{in}_d\, t'
\end{array}
$$

Figure 10: Interpretations of types

variables. We will prove strong normalization for classically typed terms using a version of Krivine's classical realizability [20]. We define three interpretations of types in Figure 10. The definition is by mutual induction, and can easily be seen to be well-founded, as the definition of $[\![A]\!]^{+}$ invokes the definition of $[\![A]\!]^{-}$ with the same type, which in turn invokes the definition of $[\![A]\!]^{+c}$ with the same type; and the definition of $[\![A]\!]^{+c}$ may invoke either of the other definitions at a strictly smaller type. The reader familiar with such proofs will also recognize the debt owed to Girard [12].

**Lemma 52** (Step interpretations)**.** If $t \in [\![A]\!]^{+}$ and $t \rightsquigarrow t'$, then $t' \in [\![A]\!]^{+}$; and similarly if $t \in [\![A]\!]^{-}$ or $t \in [\![A]\!]^{+c}$.

*Proof.* The proof is by a mutual well-founded induction. Assume $t \in [\![A]\!]^{+}$ and $t \rightsquigarrow t'$. We must show $t' \in [\![A]\!]^{+}$. For this, it suffices to assume $y \in \mathit{Var}$ and $t'' \in [\![A]\!]^{-}$, and show $\nu\, y.t' \bullet t'' \in \mathbf{SN}$. From the assumption that $t \in [\![A]\!]^{+}$, we have

$$\nu\, y.t \bullet t'' \in \mathbf{SN}$$

which indeed implies that

$$\nu\, y.t' \bullet t'' \in \mathbf{SN}$$

A similar argument applies if $t \in [\![A]\!]^{-}$.

For the last part of the lemma, assume $t \in [\![A]\!]^{+c}$ with $t \rightsquigarrow t'$, and show $t' \in [\![A]\!]^{+c}$. The only possible cases are the following, where $t \notin \mathit{Vars}$.

If $A \equiv A_1 \rightarrow_{+} A_2$, then $t$ is of the form $\lambda x.t_a$ for some $x$ and $t_a$, where for all $t_b \in [\![A_1]\!]^{+}$, we have $[t_b/x]t_a \in [\![A_2]\!]^{+}$. Since $t \rightsquigarrow t'$, $t'$ must be $\lambda x.t'_a$ for some $t'_a$ with $t_a \rightsquigarrow t'_a$. It suffices now to assume an arbitrary $t_b \in [\![A_1]\!]^{+}$, and show $[t_b/x]t'_a \in [\![A_2]\!]^{+}$. But $[t_b/x]t_a \rightsquigarrow [t_b/x]t'_a$ follows from $t_a \rightsquigarrow t'_a$, so by our IH, we have $[t_b/x]t'_a \in [\![A_2]\!]^{+}$, as required.

If $A \equiv A_1 \rightarrow_{-} A_2$, then $t$ is of the form $\langle t_1, t_2 \rangle$ for some $t_1 \in [\![A_1]\!]^{-}$ and $t_2 \in [\![A_2]\!]^{+}$; and $t' \equiv \langle t'_1, t'_2 \rangle$ where either $t'_1 \equiv t_1$ and $t_2 \rightsquigarrow t'_2$ or else $t_1 \rightsquigarrow t'_1$ and $t'_2 \equiv t_2$. Either way, we have $t'_1 \in [\![A_1]\!]^{-}$ and $t'_2 \in [\![A_2]\!]^{+}$ by our IH, so we have $\langle t'_1, t'_2 \rangle \in [\![A_1 \rightarrow_{-} A_2]\!]^{+c}$ as required.

The other cases for $A \equiv A_1 \wedge_p A_2$ are similar to the previous one. □

**Lemma 53** (SN interpretations)**.**

1. $[\![A]\!]^{+} \subseteq \mathbf{SN}$
2. $\mathit{Vars} \subseteq [\![A]\!]^{-}$
3. $[\![A]\!]^{-} \subseteq \mathbf{SN}$
4. $[\![A]\!]^{+c} \subseteq \mathbf{SN}$

*Proof.* The proof holds by mutual well-founded induction on the pair $(A, n)$, where $n$ is the number of the proposition in the statement of the lemma; the well-founded ordering in question is the lexicographic combination of the structural ordering on types (for $A$) and the ordering $1 > 2 > 4 > 3$ (for $n$). For the full proof see Appendix C.6. □

**Definition 54** (Interpretation of contexts)**.** $[\![\Gamma]\!]$ is the set of substitutions $\sigma$ such that for all $x : p\ A \in \Gamma$, $\sigma(x) \in [\![A]\!]^p$.

**Lemma 55** (Canonical positive is positive)**.** $[\![A]\!]^{+c} \subseteq [\![A]\!]^+$

*Proof.* Assume $t \in [\![A]\!]^{+c}$ and show $t \in [\![A]\!]^+$. For the latter, assume arbitrary $x \in \mathit{Vars}$ and $t' \in [\![A]\!]^-$, and show $\nu\, x.t \bullet t' \in \mathbf{SN}$. This follows immediately from the assumption that $t' \in [\![A]\!]^-$. □

**Theorem 56** (Soundness)**.** If $\Gamma \vdash_c t : p\ A$ then for all $\sigma \in [\![\Gamma]\!]$, $\sigma t \in [\![A]\!]^p$.

*Proof.* The proof holds by induction on the derivation of $\Gamma \vdash_c t : p\ A$. For the full proof see Appendix C.7. □

**Corollary 57** (Strong Normalization)**.** If $G; \Gamma \vdash t : p\ A \mathbin{@} n$, then $t \in \mathbf{SN}$.

*Proof.* This follows easily by putting together Theorems 51 and 56, with Lemma 53. □

**Corollary 58** (Cut Elimination)**.** If $G; \Gamma \vdash t : p\ A \mathbin{@} n$, then there is normal $t'$ with $t \rightsquigarrow^* t'$ and $t'$ containing only cut terms of the form $\nu\, x.y \bullet t$ or $\nu\, x.t \bullet y$, for $y$ a variable.

**Lemma 59** (Local Confluence)**.** The reduction relation of Figure 8 is locally confluent.

*Proof.* We may view the reduction rules as higher-order pattern rewrite rules. It is easy to confirm that all critical pairs (e.g., between RBetaR and the rules RImp, RImpBar, RAnd1, RAndBar1, RAnd2, and RAndBar2) are joinable. Local confluence then follows by the higher-order critical pair lemma [26]. □

**Theorem 60** (Confluence for Typable Terms)**.** The reduction relation restricted to terms typable in DTT is confluent.

*Proof.* Suppose $G; \Gamma \vdash t : p\ A \mathbin{@} n$ for some $G$, $\Gamma$, $p$, and $A$. By Lemma 50, any reductions in the unrestricted reduction relation from $t$ are also in the reduction relation restricted to typable terms. The result now follows from Newman's Lemma, using Lemma 59 and Theorem 57. □

## 7. Conclusion

We have presented a new type theory for bi-intuitionistic logic. We began with a compact dualized formulation of the logic, Dualized Intuitionistic Logic (DIL), and showed soundness with respect to a standard Kripke semantics (in Agda), and completeness with respect to Pinto and Uustalu's system L. We then presented Dualized Type Theory (DTT), and showed type preservation, strong normalization, and confluence for typable terms. Future work includes further additions to DTT, for example with polymorphism and inductive types. It would also be interesting to obtain a Canonicity Theorem as in [34], identifying some set of types where closed normal forms are guaranteed to be canonical values (as canonicity fails in general in DIL/DTT, as in other bi-intuitionistic systems).

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## Appendix A. Proofs from Section 4.2: Completeness of DIL

A.1. **Proof of Lemma 17.** Suppose $G;\Gamma,\bar{p}\ A \mathbin{@} n \vdash p\ B \mathbin{@} n$ is derivable. By weakening we know $G;\Gamma,\bar{p}\,(A \wedge_{\bar{p}} B) \mathbin{@} n, \bar{p}\ B \mathbin{@} n, \bar{p}\ A \mathbin{@} n \vdash p\ B \mathbin{@} n$. Then $G;\Gamma \vdash p\,(A \wedge_{\bar{p}} B) \mathbin{@} n$ is derivable as follows:

$$
\dfrac{\bar{p}\,(A \wedge_{\bar{p}} B) \mathbin{@} n \in \Gamma, \bar{p}\,(A \wedge_{\bar{p}} B) \mathbin{@} n \qquad \dfrac{\dfrac{D_1 \qquad D_2}{G;\Gamma,\bar{p}\,(A \wedge_{\bar{p}} B) \mathbin{@} n \vdash p\ B \mathbin{@} n}\ \textsc{Cut}}{G;\Gamma,\bar{p}\,(A \wedge_{\bar{p}} B) \mathbin{@} n \vdash p\,(A \wedge_{\bar{p}} B) \mathbin{@} n}\ \textsc{AndBar}}{G;\Gamma \vdash p\,(A \wedge_{\bar{p}} B) \mathbin{@} n}\ \textsc{axCut}
$$

where we have the following subderivations:

$D_0$ :

$$
\dfrac{\dfrac{}{G;\Gamma,\bar{p}\,(A \wedge_{\bar{p}} B) \mathbin{@} n, \bar{p}\ B \mathbin{@} n, p\ A \mathbin{@} n \vdash p\ A \mathbin{@} n}\ \textsc{ax}}{G;\Gamma,\bar{p}\,(A \wedge_{\bar{p}} B) \mathbin{@} n, \bar{p}\ B \mathbin{@} n, p\ A \mathbin{@} n \vdash p\,(A \wedge_{\bar{p}} B) \mathbin{@} n}\ \textsc{AndBar}
$$

$D_1$ :

$$
\dfrac{\bar{p}\ B \mathbin{@} n \in \Gamma, \bar{p}\,(A \wedge_{\bar{p}} B) \mathbin{@} n, \bar{p}\ B \mathbin{@} n, \bar{p}\ A \mathbin{@} n \qquad G;\Gamma,\bar{p}\,(A \wedge_{\bar{p}} B) \mathbin{@} n, \bar{p}\ B \mathbin{@} n, \bar{p}\ A \mathbin{@} n \vdash p\ B \mathbin{@} n}{G;\Gamma,\bar{p}\,(A \wedge_{\bar{p}} B) \mathbin{@} n, \bar{p}\ B \mathbin{@} n \vdash p\ A \mathbin{@} n}\ \textsc{axCut}
$$

$D_2$ :

$$
\dfrac{\bar{p}\,(A \wedge_{\bar{p}} B) \mathbin{@} n \in \Gamma, \bar{p}\,(A \wedge_{\bar{p}} B) \mathbin{@} n, \bar{p}\ B \mathbin{@} n, p\ A \mathbin{@} n \qquad D_0}{G;\Gamma,\bar{p}\,(A \wedge_{\bar{p}} B) \mathbin{@} n, \bar{p}\ B \mathbin{@} n \vdash \bar{p}\ A \mathbin{@} n}\ \textsc{axCut}
$$

A.2. **Proof of Lemma 22: Raising the Lower Bound.** This is a proof by induction on the form of $G, G_1 \vdash m \preccurlyeq^*_{p'} m'$.

Case.

$$\frac{}{G', m \preccurlyeq_{p'} m', G'' \vdash m \preccurlyeq^*_{p'} m'}\ \text{AX}$$

Note that it is the case that $G', m \preccurlyeq_{p'} m', G'' \equiv G, G_1$. If $m \preccurlyeq_{p'} m' \in G$, then we obtain our result, so suppose $m \preccurlyeq_{p'} m' \in G_1$. Suppose $p \equiv p'$. Now if $m \not\equiv n_1$, then clearly, we obtain our result. Consider the case where $m \equiv n_1$. Then it suffices to show $G, \mathsf{raise}\,(n_1, n_2, G'_1), n_2 \preccurlyeq_p m', \mathsf{raise}\,(n_1, n_2, G''_1) \vdash n_1 \preccurlyeq^*_p m'$ where $G_1 \equiv G'_1, n_1 \preccurlyeq_p m', G''_1$. This holds by the following derivation:

$$\frac{G \vdash n_1 \preccurlyeq^*_p n_2 \qquad \dfrac{}{G, \mathsf{raise}\,(n_1, n_2, G'_1), n_2 \preccurlyeq_p m', \mathsf{raise}\,(n_1, n_2, G''_1) \vdash n_2 \preccurlyeq^*_p m'}\ \text{REL\_AX}}{G, \mathsf{raise}\,(n_1, n_2, G'_1), n_2 \preccurlyeq_p m', \mathsf{raise}\,(n_1, n_2, G''_1) \vdash n_1 \preccurlyeq^*_p m'}\ \text{REL\_TRANS}$$

Now suppose $p' \equiv \bar{p}$. if $m' \not\equiv n_1$, then clearly, we obtain our result. Consider the case where $m' \equiv n_1$. Then it suffices to show $G, \mathsf{raise}\,(n_1, n_2, G'_1), m \preccurlyeq_{\bar{p}} n_2, \mathsf{raise}\,(n_1, n_2, G''_1) \vdash m \preccurlyeq^*_{\bar{p}} n_1$ where $G_1 \equiv G'_1, m \preccurlyeq_{\bar{p}} n_1, G''_1$. This holds by the following derivation:

$$\frac{\dfrac{}{G, \mathsf{raise}\,(n_1, n_2, G'_1), m \preccurlyeq_{\bar{p}} n_2, \mathsf{raise}\,(n_1, n_2, G''_1) \vdash m \preccurlyeq^*_{\bar{p}} n_2}\ \text{REL\_AX} \qquad \dfrac{G \vdash n_1 \preccurlyeq^*_p n_2}{G \vdash n_2 \preccurlyeq^*_{\bar{p}} n_1}\ \text{REL\_FLIP}}{G, \mathsf{raise}\,(n_1, n_2, G'_1), m \preccurlyeq_{\bar{p}} n_2, \mathsf{raise}\,(n_1, n_2, G''_1) \vdash m \preccurlyeq^*_{\bar{p}} n_1}\ \text{REL\_TRANS}$$

Case.

$$\frac{}{G, G_1 \vdash m \preccurlyeq^*_{p'} m}\ \text{REFL}$$

Note that in this case $m' \equiv m$. Our result follows from simply an application of the REL_REFL rule.

Case.

$$\frac{G, G_1 \vdash m \preccurlyeq^*_{p'} m'' \qquad G, G_1 \vdash m'' \preccurlyeq^*_{p'} m'}{G, G_1 \vdash m \preccurlyeq^*_{p'} m'}\ \text{REL\_TRANS}$$

This case holds by two applications of the induction hypothesis followed by applying the REL_TRANS rule.

Case.

$$\frac{G, G_1 \vdash m' \preccurlyeq^*_{\bar{p}'} m}{G, G_1 \vdash m \preccurlyeq^*_{p'} m'}\ \text{FLIP}$$

This case holds by an application of the induction hypothesis followed by applying the REL_FLIP rule.

A.3. **Proof of Lemma 24: Raising the Lower Bound Logically.** This is a proof by induction on the form of $G, G_1, G'; \Gamma \vdash p\, A \,@\, n$. We assume with out loss of generality that $n_1 \in |G_1|$, and that $n_1 \not\equiv n_2$. If this is not the case then $\mathsf{raise}\,(n_1, n_2, G_1) = G_1$, and the result holds trivially.

Case.

$$\frac{G, G_1, G' \vdash n' \preccurlyeq^*_p n}{G, G_1, G'; \Gamma, p\, A \,@\, n' \vdash p\, A \,@\, n}\ \text{AX}$$

Clearly, if $G, G_1, G' \vdash n' \preccurlyeq^*_p n$, then $G, G', G_1 \vdash n' \preccurlyeq^*_p n$. Thus, this case follows by raising the lower bound (Lemma 22), and applying the AX rule.

Case.

$$\frac{}{G, G_1, G'; \Gamma \vdash p\ \langle p \rangle\ @\ n}\ \textsc{unit}$$

Trivial.

Case.

$$\frac{G, G_1, G'; \Gamma \vdash p\ A_1\ @\ n \qquad G, G_1, G'; \Gamma \vdash p\ A_2\ @\ n}{G, G_1, G'; \Gamma \vdash p\ (A_1 \wedge_p A_2)\ @\ n}\ \textsc{and}$$

This case holds by two applications of the induction hypothesis, and then applying the AND rule.

Case.

$$\frac{G, G_1, G'; \Gamma \vdash p\ A_d\ @\ n}{G, G_1, G'; \Gamma \vdash p\ (A_1 \wedge_{\bar{p}} A_2)\ @\ n}\ \textsc{andBar}$$

Similar to the previous case.

Case.

$$\frac{\begin{array}{l} n' \notin |G, G_1, G'|, |\Gamma| \\ (G, G_1, G', n \preccurlyeq_p n'); \Gamma, p\ A_1\ @\ n' \vdash p\ A_2\ @\ n' \end{array}}{G, G_1, G'; \Gamma \vdash p\ (A_1 \to_p A_2)\ @\ n}\ \textsc{imp}$$

Since we know $n_1 \not\equiv n_2$, then by Lemma 23 we know $n_1, n_2 \in |G, G'|$. Thus, $n' \not\equiv n_1 \not\equiv n_2$. Now by the induction hypothesis we know $(G, \mathsf{raise}\,(n_1, n_2, G_1), G', n \preccurlyeq_p n'); \Gamma, p\ A_1\ @\ n' \vdash p\ A_2\ @\ n'$. This case then follows by the application of the IMP rule to the former.

Case.

$$\frac{\begin{array}{l} G, G_1, G' \vdash n \preccurlyeq^*_{\bar{p}} n' \\ G, G_1, G'; \Gamma \vdash \bar{p}\ A_1\ @\ n' \qquad G, G_1, G'; \Gamma \vdash p\ A_2\ @\ n' \end{array}}{G, G_1, G'; \Gamma \vdash p\ (A_1 \to_{\bar{p}} A_2)\ @\ n}\ \textsc{impBar}$$

Clearly, $G, G_1, G' \vdash n \preccurlyeq^*_{\bar{p}} n'$ implies $G, G', G_1 \vdash n \preccurlyeq^*_{\bar{p}} n'$, and by raising the lower bound (Lemma 22) we know $G, G', \mathsf{raise}\,(n_1, n_2, G_1) \vdash n \preccurlyeq^*_{\bar{p}} n'$ which implies $G, \mathsf{raise}\,(n_1, n_2, G_1), G' \vdash n \preccurlyeq^*_{\bar{p}} n'$. Thus, this case follows from applying IMPBAR to the application of the induction hypothesis to each premise and $G, \mathsf{raise}\,(n_1, n_2, G_1), G' \vdash n \preccurlyeq^*_{\bar{p}} n'$.

Case.

$$\frac{p\ T'\ @\ n' \in \Gamma \qquad G, G_1, G'; \Gamma, \bar{p}\ T\ @\ n \vdash \bar{p}\ T'\ @\ n'}{G, G_1, G'; \Gamma \vdash p\ T\ @\ n}\ \textsc{axCut}$$

This case follows by a simple application of the induction hypothesis, and then reapplying the rule.

Case.

$$\frac{\bar{p}\ T' \mathbin{@} n' \in \Gamma \qquad G, G_1, G'; \Gamma, \bar{p}\ T \mathbin{@} n \vdash p\ T' \mathbin{@} n'}{G, G_1, G'; \Gamma \vdash p\ T \mathbin{@} n}\ \textsc{axCutBar}$$

Similar to the previous case.

A.4. **Proof of Lemma 25: General Monotonicity.** This is a proof by induction on the form of $G; \bar{p_1}\ A_1 \mathbin{@} n_1, \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ B \mathbin{@} m$. We assume without loss of generality that all of $n_1, n'_1 \ldots, n_i, n'_i$ are unique. Thus, they are all members of $|G|$ by Lemma 23.

Case.

$$\frac{G \vdash n_j \preccurlyeq^*_{\bar{p_j}} m}{G; \bar{p_1}\ A_1 \mathbin{@} n_1, \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash \bar{p_j}\ A_j \mathbin{@} m}\ \textsc{ax}$$

It must be the case that $p\, B \mathbin{@} m \equiv \bar{p_j}\ A_j \mathbin{@} m$ for some $1 \leq j \leq i$. In addition, we know $G \vdash n_j \preccurlyeq^*_{p_j} n'_j$, $G \vdash n_j \preccurlyeq^*_{\bar{p_j}} m$, and $G \vdash m \preccurlyeq^*_{\bar{p_j}} m'$. It suffices to show $G; \bar{p_1} A_1 @ n'_1, \ldots, \bar{p_i} A_i @ n'_i \vdash \bar{p_j}\ A_j \mathbin{@} m'$, but to obtain this result it suffices to show that $G \vdash n'_j \preccurlyeq_{\bar{p_j}} m'$, but this holds by first using REL_FLIP to obtain $G \vdash n'_j \preccurlyeq^*_{\bar{p_j}} n_j$ followed by two applications of transitivity.

Case.

$$\frac{}{G; \bar{p_1}\ A_1 \mathbin{@} n_1, \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ \langle p \rangle \mathbin{@} m_1}\ \textsc{unit}$$

Trivial.

Case.

$$\frac{\begin{array}{c} G; \bar{p_1}\ A_1 \mathbin{@} n_1, \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ B_1 \mathbin{@} m \\ G; \bar{p_1}\ A_1 \mathbin{@} n_1, \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ B_2 \mathbin{@} m \end{array}}{G; \bar{p_1}\ A_1 \mathbin{@} n_1, \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ (B_1 \wedge_p B_2) \mathbin{@} m}\ \textsc{and}$$

This case follows easily by applying the induction hypothesis to each premise and then applying the AND rule.

Case.

$$\frac{G; \bar{p_1}\ A_1 \mathbin{@} n_1, \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ B_d \mathbin{@} m}{G; \bar{p_1}\ A_1 \mathbin{@} n_1, \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ (B_1 \wedge_{\bar{p}} B_2) \mathbin{@} m}\ \textsc{andBar}$$

This case follows easily by the induction hypothesis and then applying ANDBAR.

Case.

$$\frac{\begin{array}{l} n' \notin |G|, |\bar{p_1}\ A_1 \mathbin{@} n_1, \ldots, \bar{p_i}\ A_i \mathbin{@} n_i| \\ (G, m_1 \preccurlyeq_p n'); \bar{p_1}\ A_1 \mathbin{@} n_1, \ldots, \bar{p_i}\ A_i \mathbin{@} n_i, p\ B_1 \mathbin{@} n' \vdash p\ B_2 \mathbin{@} n' \end{array}}{G; \bar{p_1}\ A_1 \mathbin{@} n_1, \ldots, \bar{p_i}\ A_i \mathbin{@} n_i \vdash p\ (B_1 \rightarrow_p B_2) \mathbin{@} m}\ \textsc{imp}$$

We know by assumption $G \vdash n_1 \preccurlyeq^*_{p_1} n'_1, \ldots, G \vdash n_i \preccurlyeq^*_{p_i} n'_i$, and by graph weakening (Lemma 20) $G, m \preccurlyeq_p n' \vdash n_1 \preccurlyeq^*_{p_1} n'_1, \ldots, G, m \preccurlyeq_p n' \vdash n_i \preccurlyeq^*_{p_i} n'_i$. We also know by applying the REL_REFL rule that $G, m \preccurlyeq_p n' \vdash n' \preccurlyeq^*_{\bar{p}} n'$ and $G, m \preccurlyeq_p n' \vdash n' \preccurlyeq^*_p n'$. Thus, by the induction hypothesis we know $(G, m \preccurlyeq_p n'); \bar{p_1} A_1 @ n'_1, \ldots, \bar{p_i} A_i @ n'_i, p\, B_1 \mathbin{@} n' \vdash p\, B_2 \mathbin{@} n'$. Now we can raise the lower bound logically (Lemma 24) with $G_1 \equiv m \preccurlyeq_p n'$ and the assumption $G \vdash m \preccurlyeq^*_p m'$ to

obtain
$(G, \mathsf{raise}\,(m, m', m \preccurlyeq_p n')); \bar{p_1} A_1 @ n'_1, \ldots, \bar{p_i} A_i @ n'_i, p\, B_1 @ n' \vdash p\, B_2 @ n'$, but this is equivalent to $(G, m \preccurlyeq_p n'); \bar{p_1} A_1 @ n'_1, \ldots, \bar{p_i} A_i @ n'_i, p\, B_1 @ n' \vdash p\, B_2 @ n'$. Finally, using the former, we obtain our result by applying the IMP rule.

Case.

$$\frac{\begin{array}{l} G \vdash m \preccurlyeq^*_{\bar{p}} n' \\ G; \bar{p_1}\, A_1 @ n_1, \ldots, \bar{p_i}\, A_i @ n_i \vdash \bar{p}\, B_1 @ n' \\ G; \bar{p_1}\, A_1 @ n_1, \ldots, \bar{p_i}\, A_i @ n_i \vdash p\, B_2 @ n' \end{array}}{G; \bar{p_1}\, A_1 @ n_1, \ldots, \bar{p_i}\, A_i @ n_i \vdash p\,(B_1 \to_{\bar{p}} B_2) @ m}\ \textsc{impBar}$$

We can easily derive $G \vdash m' \preccurlyeq^*_{\bar{p}} n'$ as follows:

$$\frac{\dfrac{\dfrac{G \vdash m \preccurlyeq^*_{\bar{p}} n'}{G \vdash n' \preccurlyeq^*_{p} m}\ \textsc{rel\_flip} \qquad G \vdash m \preccurlyeq^*_{p} m'}{G \vdash n' \preccurlyeq^*_{p} m'}\ \textsc{rel\_trans}}{G \vdash m' \preccurlyeq^*_{\bar{p}} n'}\ \textsc{rel\_flip}$$

This case then follows by applying the induction hypothesis twice to both $G; \bar{p_1}\, A_1 @ n_1, \ldots, \bar{p_i}\, A_i @ n_i \vdash \bar{p}\, B_1 @ n'$ and $G; \bar{p_1}\, A_1 @ n_1, \ldots, \bar{p_i}\, A_i @ n_i \vdash p\, B_2 @ n'$ using the assumptions $G \vdash n_1 \preccurlyeq^*_{p_1} n'_1, \ldots, G \vdash n_i \preccurlyeq^*_{p_i} n'_i$, and the fact that we know $G \vdash n' \preccurlyeq^*_{p} n'$ and $G \vdash n' \preccurlyeq^*_{\bar{p}} n'$.

Case.

$$\frac{\begin{array}{l} \bar{p_j}\, A_j @ n_j \in (\bar{p_1}\, A_1 @ n_1, \ldots, \bar{p_i}\, A_i @ n_i) \\ G; \bar{p_1}\, A_1 @ n_1, \ldots, \bar{p_i}\, A_i @ n_i, \bar{p}\, B @ m \vdash p_j\, A_j @ n_j \end{array}}{G; \bar{p_1}\, A_1 @ n_1, \ldots, \bar{p_i}\, A_i @ n_i \vdash p\, B @ m}\ \textsc{axCut}$$

We know by assumption that $G \vdash n_1 \preccurlyeq^*_{p_1} n'_1, \ldots, G \vdash n_i \preccurlyeq^*_{p_i} n'_i$, and $G \vdash m \preccurlyeq^*_{p} m'$. In particular, we know $G \vdash n_j \preccurlyeq^*_{p_j} n'_j$. It is also the case that if $\bar{p_j}\, A_j @ n_j \in (\bar{p_1}\, A_1 @ n_1, \ldots, \bar{p_i}\, A_i @ n_i)$, then $\bar{p_j}\, A_j @ n'_j \in (\bar{p_1} A_1 @ n', \ldots, \bar{p_i} A_i @ n'_i)$. This case then follows by applying the induction hypothesis to $G; \bar{p_1}\, A_1 @ n_1, \ldots, \bar{p_i}\, A_i @ n_i, \bar{p}\, B @ m \vdash p_j\, A_j @ n_j$, to obtain, $G; \bar{p_1} A_1 @ n'_1, \ldots, \bar{p_i} A_i @ n'_i, \bar{p}\, B @ m'_1 \vdash p_j\, A_j @ n'_j$, followed by applying the AXCUT rule.

Case.

$$\frac{\begin{array}{l} \bar{p_j}\, A_j @ n_j \in (\bar{p_1}\, A_1 @ n_1, \ldots, \bar{p_i}\, A_i @ n_i) \\ G; \bar{p_1}\, A_1 @ n_1, \ldots, \bar{p_i}\, A_i @ n_i, \bar{p}\, B @ m \vdash p_j\, A_j @ n_j \end{array}}{G; \bar{p_1}\, A_1 @ n_1, \ldots, \bar{p_i}\, A_i @ n_i \vdash p\, B @ m}\ \textsc{axCutBar}$$

Similar to the previous case.

A.5. **Proof of Lemma 31: Containment of L in DIL.** This is a proof by induction on the form of the sequent $\Gamma \vdash_G \Delta$.

Case.

$$\frac{\Gamma \vdash_{G,(n,n)} \Delta}{\Gamma \vdash_G \Delta}\ \textsc{refl}$$

We know by the induction hypothesis that every activation of $\Gamma \vdash_{G,(n,n)} \Delta$ is derivable. Suppose that $\mathsf{D}(G,(n,n)); \mathsf{D}(\Gamma)^+, \Gamma' \vdash +A \mathbin{@} n$ is an arbitrary activation, where $\mathsf{D}(\Delta)^- \equiv \mathsf{D}(\Delta_1)^-, -A \mathbin{@} n, \mathsf{D}(\Delta_2)^-$ and $\Gamma' \equiv \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^-$. This is equivalent to $\mathsf{D}(G), n \preccurlyeq_+ n; \mathsf{D}(\Gamma)^+, \Gamma' \vdash +A \mathbin{@} n$, and by the admissible rule for reflexivity (Lemma 15) we have $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \Gamma' \vdash +A \mathbin{@} n$.

Case.

$$\dfrac{\begin{array}{l} n_1 G n_2 \\ n_2 G n_3 \\ \Gamma \vdash_{G,(n_1,n_3)} \Delta \end{array}}{\Gamma \vdash_G \Delta}\ \textsc{Trans}$$

We know by the induction hypothesis that every activation of $\Gamma \vdash_{G,(n_1,n_3)} \Delta$ is derivable. Suppose that $\mathsf{D}(G,(n_1,n_3)); \mathsf{D}(\Gamma)^+, \Gamma' \vdash +A \mathbin{@} n$ is an arbitrary activation, where $\mathsf{D}(\Delta)^- \equiv \mathsf{D}(\Delta_1)^-, -A \mathbin{@} n, \mathsf{D}(\Delta_2)^-$ and $\Gamma' \equiv \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^-$. This sequent is equivalent to $\mathsf{D}(G), n_1 \preccurlyeq_+ n_3; \mathsf{D}(\Gamma)^+, \Gamma' \vdash +A \mathbin{@} n$. Furthermore, it is clear by definition that if $n_1 G n_2$ and $n_2 G n_3$, then $n_1 \preccurlyeq_+ n_2 \in \mathsf{D}(G)$ and $n_2 \preccurlyeq_+ n_3 \in \mathsf{D}(G)$. Thus, by the admissible rule for transitivity (Lemma 16) we have $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \Gamma' \vdash +A \mathbin{@} n$, and we obtain our result.

Case.

$$\dfrac{}{\Gamma, n : A \vdash_G n : A, \Delta}\ \textsc{hyp}$$

It suffices to show that every activation of $\Gamma, n : A \vdash_G n : A, \Delta$ is derivable. Clearly, $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, +\mathsf{D}(A)@n, \mathsf{D}(\Delta)^- \vdash +\mathsf{D}(A)@n$ is a activation of $\Gamma, n : A \vdash_G n : A, \Delta$. In addition, it is derivable:

$$\dfrac{\dfrac{\dfrac{}{\mathsf{D}(G) \vdash n \preccurlyeq^*_+ n}\ \textsc{Refl}}{\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^-, +\mathsf{D}(A) \mathbin{@} n \vdash +\mathsf{D}(A) \mathbin{@} n}\ \textsc{Ax}}{\mathsf{D}(G); \mathsf{D}(\Gamma)^+, +\mathsf{D}(A) \mathbin{@} n, \mathsf{D}(\Delta)^- \vdash +\mathsf{D}(A) \mathbin{@} n}\ \textsc{Exchange}$$

In the previous derivation we make use of the exchange rule which is admissible by Lemma 18.

Now consider any other activation $\mathsf{D}(G); \Gamma' \vdash +\mathsf{D}(B) \mathbin{@} n'$. It must be the case that $\Gamma' = \mathsf{D}(\Gamma)^+, +\mathsf{D}(A) \mathbin{@} n, \mathsf{D}(\Delta_1)^-, -\mathsf{D}(A) \mathbin{@} n, \mathsf{D}(\Delta_2)^-$ for some $\Delta_1$ and $\Delta_2$. This sequent is then derivable as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{D}(G) \vdash n \preccurlyeq^*_+ n}\ \textsc{Refl}}{\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^-, -\mathsf{D}(B) \mathbin{@} n', +\mathsf{D}(A) \mathbin{@} n \vdash +\mathsf{D}(A) \mathbin{@} n}\ \textsc{Ax}}{\mathsf{D}(G); \mathsf{D}(\Gamma)^+, +A \mathbin{@} n, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^-, -B \mathbin{@} n' \vdash +\mathsf{D}(A) \mathbin{@} n}\ \textsc{Exchange}}{\mathsf{D}(G); \mathsf{D}(\Gamma)^+, +\mathsf{D}(A) \mathbin{@} n, \mathsf{D}(\Delta_1)^-, -\mathsf{D}(A) \mathbin{@} n, \mathsf{D}(\Delta_2)^- \vdash +\mathsf{D}(B) \mathbin{@} n'}\ \textsc{Left-to-Right}$$

Thus, we obtain our result.

Case.

$$\dfrac{\begin{array}{l} n_1 G n_2 \\ \Gamma, n_1 : A, n_2 : A \vdash_G \Delta \end{array}}{\Gamma, n_1 : A \vdash_G \Delta}\ \textsc{monL}$$

Certainly, if $n_1 G n_2$, then $n_1 \preccurlyeq_+ n_2 \in \mathsf{D}(G)$. We know by the induction hypothesis that all activations of $\Gamma, n_1 : A, n_2 : A \vdash_G \Delta$ are derivable. Suppose $\mathsf{D}(G); \Gamma' \vdash +B @ n$ is an arbitrary activation. Then it must be the case that $\Gamma' \equiv \mathsf{D}(\Gamma)^+, +\mathsf{D}(A) @ n_1, +\mathsf{D}(A) @ n_2, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^-$, where $\mathsf{D}(\Delta)^- \equiv \mathsf{D}(\Delta_1)^-, -B @ n, \mathsf{D}(\Delta_2)^-$. Now we apply the monoL admissible rule (Lemma 27) to obtain $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, +\mathsf{D}(A) @ n_1, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^- \vdash +B @ n$, which is an arbitrary activation of $\Gamma, n_1 : A \vdash_G \Delta$.

Case.

$$\frac{\begin{array}{l} n_1 G n_2 \\ \Gamma \vdash_G n_1 : A, n_2 : A, \Delta \end{array}}{\Gamma \vdash_G n_2 : A, \Delta}\ \textsc{monR}$$

If $n_1 G n_2$, then $n_1 \preccurlyeq_+ n_2 \in \mathsf{D}(G)$. We know by the induction hypothesis that all activations of $\Gamma \vdash_G n_1 : A, n_2 : A, \Delta$ are derivable. In particular, the activation (modulo exchange (Lemma 18)) $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^-, -\mathsf{D}(A) @ n_1 \vdash +\mathsf{D}(A) @ n_2$ is derivable. It suffices to show that $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash +\mathsf{D}(A) @ n_2$. This follows from the monoR admissible rule (Lemma 28). Finally, any other activation of $\Gamma \vdash_G n_2 : A, \Delta$ can be activated into $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash +\mathsf{D}(A) @ n_2$ (Lemma 14). Thus, we obtain our result.

Case.

$$\frac{\Gamma \vdash_G \Delta}{\Gamma, n' : \top \vdash_G \Delta}\ \textsc{trueL}$$

We know by the induction hypothesis that all activations of $\Gamma \vdash_G \Delta$ are derivable. Suppose $\mathsf{D}(G); \Gamma' \vdash +\mathsf{D}(A) @ n$ is an arbitrary activation of $\Gamma \vdash_G \Delta$. Then it must be the case that $\Gamma' = \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^-$, where $\mathsf{D}(\Delta)^- \equiv \mathsf{D}(\Delta_1)^-, -\mathsf{D}(A) @ n, \mathsf{D}(\Delta_2)^-$. Now by weakening (Lemma 13) we know $\mathsf{D}(G); \Gamma', +\langle + \rangle @ n' \vdash +\mathsf{D}(A) @ n$, and by exchange (Lemma 18)
$\mathsf{D}(G); \mathsf{D}(\Gamma)^+, +\langle + \rangle @ n', \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^- \vdash +\mathsf{D}(A) @ n$, which is exactly an arbitrary activation of $\Gamma, n' : \top \vdash_G \Delta$.

Case.

$$\frac{}{\Gamma \vdash_G n : \top, \Delta}\ \textsc{trueR}$$

It suffices to show that every activation of $\Gamma \vdash_G n : \top, \Delta$ is derivable. Consider the activation $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash +\mathsf{D}(\top) @ n$. This is easily derivable by applying the UNIT rule. Any other activation of $\Gamma \vdash_G n : \top, \Delta$ is derivable, because $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash +\mathsf{D}(\top) @ n$ can be activated by Lemma 14.

Case.

$$\frac{}{\Gamma, n : \perp \vdash_G \Delta}\ \textsc{falseL}$$

Suppose $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, +\mathsf{D}(\perp) @ n, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^- \vdash +\mathsf{D}(A) @ n'$ is an arbitrary activation of $\Gamma, n : \perp \vdash_G \Delta$, where $\mathsf{D}(\Delta)^- \equiv \mathsf{D}(\Delta_1)^-, -\mathsf{D}(A) @ n', \mathsf{D}(\Delta_2)^-$. We can easily see that by definition $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, +\mathsf{D}(\perp) @ n, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^- \vdash +\mathsf{D}(A) @ n'$

is equivalent to $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, +\langle - \rangle \mathbin{@} n, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^- \vdash +\mathsf{D}(A) \mathbin{@} n'$. We can derive the latter as follows:

$$\dfrac{+\langle - \rangle \mathbin{@} n \in \Gamma', -\mathsf{D}(A) \mathbin{@} n' \qquad \dfrac{}{\mathsf{D}(G); \Gamma', -\mathsf{D}(A) \mathbin{@} n' \vdash -\langle - \rangle \mathbin{@} n}\ \textsc{unit}}{\mathsf{D}(G); \mathsf{D}(\Gamma)^+, +\langle - \rangle \mathbin{@} n, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^- \vdash +\mathsf{D}(A) \mathbin{@} n'}\ \textsc{axCutBar}$$

In the previous derivation $\Gamma' \equiv \mathsf{D}(\Gamma)^+, +\langle - \rangle \mathbin{@} n, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^-$. Thus, any activation of $\Gamma, n :\perp \vdash_G \Delta$ is derivable.

Case.

$$\dfrac{\Gamma \vdash_G \Delta}{\Gamma \vdash_G n' :\perp, \Delta}\ \textsc{falseR}$$

We know by the induction hypothesis that all activations of $\Gamma \vdash_G \Delta$ are derivable. Suppose $\mathsf{D}(G); \Gamma' \vdash +\mathsf{D}(A) \mathbin{@} n$ is an arbitrary activation of $\Gamma \vdash_G \Delta$. Then it must be the case that $\Gamma' = \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^-$. Now by weakening (Lemma 13) we know $\mathsf{D}(G); \Gamma', -\langle - \rangle \mathbin{@} n' \vdash +\mathsf{D}(A) \mathbin{@} n$, and by the left-to-right lemma (Lemma 14) $\mathsf{D}(G); \Gamma', -\mathsf{D}(A) \mathbin{@} n \vdash +\langle - \rangle \mathbin{@} n'$, which – modulo exchange – is equivalent to $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash +\mathsf{D}(\perp) \mathbin{@} n'$. Thus, we obtain our result.

Case.

$$\dfrac{\Gamma, n : T_1, n : T_2 \vdash_G \Delta}{\Gamma, n : T_1 \wedge T_2 \vdash_G \Delta}\ \textsc{andL}$$

We know by the induction hypothesis that all activations of $\Gamma, n : T_1, n : T_2 \vdash_G \Delta$ are derivable. In particular, we know $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, +\mathsf{D}(T_1) \mathbin{@} n, +\mathsf{D}(T_2) \mathbin{@} n, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^- \vdash +\mathsf{D}(A) \mathbin{@} n'$ where $\mathsf{D}(\Delta)^- = \mathsf{D}(\Delta_1)^-, -\mathsf{D}(A) \mathbin{@} n', \mathsf{D}(\Delta_2)^-$. Using exchange we know $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^-, +\mathsf{D}(T_1) \mathbin{@} n, +\mathsf{D}(T_2) \mathbin{@} n \vdash +\mathsf{D}(A) \mathbin{@} n'$, and by the left-to-right lemma $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^-, +\mathsf{D}(T_1) \mathbin{@} n, -\mathsf{D}(A) \mathbin{@} n' \vdash -\mathsf{D}(T_2) \mathbin{@} n$, and finally by one more application of exchange $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^-, -\mathsf{D}(A) \mathbin{@} n', +\mathsf{D}(T_1) \mathbin{@} n \vdash -\mathsf{D}(T_2) \mathbin{@} n$. At this point we know $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^-, -\mathsf{D}(A) \mathbin{@} n' \vdash -\mathsf{D}(T_1) \wedge_+ \mathsf{D}(T_2) \mathbin{@} n$ by the using the admissible $\textsc{andL}$ rule (Lemma 17). Now using left-to-right $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^-, +\mathsf{D}(T_1) \wedge_+ \mathsf{D}(T_2) \mathbin{@} n \vdash +\mathsf{D}(A) \mathbin{@} n'$ is derivable. Lastly, by exchange $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, +\mathsf{D}(T_1) \wedge_+ \mathsf{D}(T_2) \mathbin{@} n, \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^- \vdash +\mathsf{D}(A) \mathbin{@} n'$ is derivable, which is clearly an arbitrary activation of $\Gamma, n : T_1 \wedge T_2 \vdash_G \Delta$.

Case.

$$\dfrac{\begin{array}{c}\Gamma \vdash_G n : A, \Delta \\ \Gamma \vdash_G n : B, \Delta\end{array}}{\Gamma \vdash_G n : A \wedge B, \Delta}\ \textsc{andR}$$

We know by the induction hypothesis that all activations of $\Gamma \vdash_G n : A, \Delta$ and $\Gamma \vdash_G n : B, \Delta$ are derivable. In particular, $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash +\mathsf{D}(A) \mathbin{@} n$ and $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash +\mathsf{D}(B) \mathbin{@} n$ are derivable. Now by applying the $\textsc{and}$ rule we obtain $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash +\mathsf{D}(A) \wedge_+ \mathsf{D}(B) \mathbin{@} n$, which is a particular activation of $\Gamma \vdash_G n : A \wedge B, \Delta$. Finally, consider any other activation, then that sequent implies $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash +\mathsf{D}(A) \wedge_+ \mathsf{D}(B) \mathbin{@} n$ is derivable using Lemma 14. Thus, we obtain our result.

Case.

$$\frac{\begin{array}{c}\Gamma, n : A \vdash_G \Delta \\ \Gamma, n : B \vdash_G \Delta\end{array}}{\Gamma, n : A \vee B \vdash_G \Delta}\ \textsc{disjL}$$

We know by the induction hypothesis that all activations of $\Gamma, n : A \vdash_G \Delta$ and $\Gamma, n : B \vdash_G \Delta$ are derivable. So suppose

$$\mathsf{D}(G); \mathsf{D}(\Gamma)^+, + \mathsf{D}(A) @ n, \mathsf{D}(\Delta')^- \vdash + \mathsf{D}(C) @ n'$$

and

$$\mathsf{D}(G); \mathsf{D}(\Gamma)^+, + \mathsf{D}(B) @ n, \mathsf{D}(\Delta')^- \vdash + \mathsf{D}(E) @ n''$$

are particular activations, where

$$\mathsf{D}(\Delta)^- \equiv \mathsf{D}(\Delta_1)^-, - \mathsf{D}(C) @ n', \mathsf{D}(\Delta_2)^-, - \mathsf{D}(E) @ n'', \mathsf{D}(\Delta_3)^-$$

and

$$\mathsf{D}(\Delta')^- \equiv \mathsf{D}(\Delta_1)^-, \mathsf{D}(\Delta_2)^-, \mathsf{D}(\Delta_3)^-.$$

By exchange (Lemma 18) we know

$$\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta')^-, + \mathsf{D}(A) @ n \vdash + \mathsf{D}(C) @ n'$$

and

$$\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta')^-, + \mathsf{D}(B) @ n \vdash + \mathsf{D}(E) @ n''.$$

Now by the left-to-right lemma (Lemma 14) we know :

$$\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta')^-, - \mathsf{D}(C) @ n' \vdash - \mathsf{D}(A) @ n$$

and

$$\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta')^-, - \mathsf{D}(E) @ n'' \vdash - \mathsf{D}(B) @ n,$$

and by applying weakening (and exchange) we know

$$\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta')^-, - \mathsf{D}(C) @ n', - \mathsf{D}(E) @ n'' \vdash - \mathsf{D}(A) @ n$$

and

$$\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta')^-, - \mathsf{D}(C) @ n', - \mathsf{D}(E) @ n'' \vdash - \mathsf{D}(B) @ n.$$

At this point we can apply the AND rule to obtain

$$\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta')^-, - \mathsf{D}(C) @ n', - \mathsf{D}(E) @ n'' \vdash - \mathsf{D}(A) \wedge_- \mathsf{D}(B) @ n$$

to which we can apply the left-to-right lemma to and obtain $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta')^-, - \mathsf{D}(E) @ n'', + \mathsf{D}(A) \wedge_- \mathsf{D}(B) @ n \vdash + \mathsf{D}(C) @ n'$. Finally, we can apply exchange again to obtain

$$\mathsf{D}(G); \mathsf{D}(\Gamma)^+, + \mathsf{D}(A) \wedge_- \mathsf{D}(B) @ n, \mathsf{D}(\Delta')^-, - \mathsf{D}(E) @ n'' \vdash + \mathsf{D}(C) @ n',$$

which – modulo exchange – is an arbitrary activation of $\Gamma, n : A \vee B \vdash_G \Delta$. Thus, we obtain our result.

Case.

$$\frac{\Gamma \vdash_G x : T_1, x : T_2, \Delta}{\Gamma \vdash_G x : T_1 \vee T_2, \Delta}\ \textsc{disjR}$$

This case is similar to the case of andR case, except, it makes use of the andBar rule.

Case.

$$\frac{\begin{array}{l} n_1 G n_2 \\ \Gamma \vdash_G n_2 : T_1, \Delta \\ \Gamma, n_2 : T_2 \vdash_G \Delta \end{array}}{\Gamma, n_1 : T_1 \supset T_2 \vdash_G \Delta}\ \textsc{impL}$$

We know by the induction hypothesis that all activations of $\Gamma \vdash_G n_2 : T_1, \Delta$ and $\Gamma, n_2 : T_2 \vdash_G \Delta$ are derivable. In particular, we know $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash + \mathsf{D}(T_1) @ n_2$ is derivable, and so is $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash - \mathsf{D}(T_2) @ n_2$. The latter being derivable by applying the induction hypothesis followed by exchange (Lemma 18) and the left-to-right lemma (Lemma 14). We know $n_1 G n_2$ by assumption and so by Lemma 30 $\mathsf{D}(G) \vdash n_1 \preccurlyeq_+^* n_2$. Thus, by applying the impBar rule we obtain $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash - \mathsf{D}(T_1) \rightarrow_+ \mathsf{D}(T_2) @ n_1$. At this point we can apply left-to-right to the previous sequent and obtain an activation of $\Gamma, n_1 : T_1 \supset T_2 \vdash_G \Delta$. Any other activations can be used to derive $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash + \mathsf{D}(T_1) @ n_2$ and $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash - \mathsf{D}(T_2) @ n_2$, and thus, thus we obtain our result.

Case.

$$\frac{\begin{array}{l} n_2 \notin |G|, |\Gamma|, |\Delta| \\ \Gamma, n_2 : T_1 \vdash_{G \cup \{(n_1, n_2)\}} n_2 : T_2, \Delta \end{array}}{\Gamma \vdash_G n_1 : T_1 \supset T_2, \Delta}\ \textsc{impR}$$

This case follows the same pattern as the previous cases. We know by the induction hypothesis that all activations of $\Gamma, n_2 : T_1 \vdash_{G \cup \{(n_1, n_2)\}} n_2 : T_2, \Delta$ are derivable. In particular, $\mathsf{D}(G), n_1 \preccurlyeq_+ n_2; \mathsf{D}(\Gamma)^+, + \mathsf{D}(T_1) @ n_2, \mathsf{D}(\Delta)^- \vdash + \mathsf{D}(T_2) @ n_2$ is derivable. By exchange (Lemma 18)
$\mathsf{D}(G), n_1 \preccurlyeq_+ n_2; \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^-, + \mathsf{D}(T_1) @ n_2 \vdash + \mathsf{D}(T_2) @ n_2$ is derivable, and by applying the imp rule we obtain $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash + \mathsf{D}(T_1) \rightarrow_+ \mathsf{D}(T_2) @ n_1$, which is a particular activation of $\Gamma \vdash_G n_1 : T_1 \supset T_2, \Delta$. Note that in the previous application of imp we use the fact that if $n_2 \notin |G|, |\Gamma|, |\Delta|$, then $n_2 \notin |\mathsf{D}(G)|, |\mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^-|$. Lastly, any other activation of $\Gamma \vdash_G n_1 : T_1 \supset T_2, \Delta$ implies $\mathsf{D}(G); \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash + \mathsf{D}(T_1) \rightarrow_+ \mathsf{D}(T_2) @ n_1$ is derivable by the left-to-right lemma, and hence is derivable.

Case.

$$\frac{\begin{array}{l} n_1 \notin |G|, |\Gamma|, |\Delta| \\ \Gamma, n_1 : T_1 \vdash_{G \cup \{(n_1, n_2)\}} n_1 : T_2, \Delta \end{array}}{\Gamma, n_2 : T_1 \prec T_2 \vdash_G \Delta}\ \textsc{subL}$$

We know by the induction hypothesis that all activation of
$\Gamma, n_1 : T_1 \vdash_{G \cup \{(n_1, n_2)\}} n_1 : T_2, \Delta$ are derivable. In particular,
$\mathsf{D}(G), n_1 \preccurlyeq_+ n_2; \mathsf{D}(\Gamma)^+, + \mathsf{D}(T_1) @ n_1, \mathsf{D}(\Delta)^- \vdash + \mathsf{D}(T_2) @ n_1$ is derivable. By exchange (Lemma 18) $\mathsf{D}(G), n_1 \preccurlyeq_+ n_2; \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^-, + \mathsf{D}(T_1) @ n_1 \vdash + \mathsf{D}(T_2) @ n_1$ is derivable. Now by the left-to-right lemma we know
$\mathsf{D}(G), n_1 \preccurlyeq_+ n_2; \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^-, - \mathsf{D}(T_2) @ n_1 \vdash - \mathsf{D}(T_1) @ n_1$, and by assumption

we know $y \notin |G|, |\Gamma|, |\Delta|$ which implies
$n_1 \notin |\mathsf{D}(G)|, |\mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^-|$ is derivable. Thus, by applying the IMP rule we know $\mathsf{D}(G), n_1 \preccurlyeq_+ n_2; \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash -\mathsf{D}(T_2) \rightarrow_- \mathsf{D}(T_1) \mathbin{@} n_2$ is derivable. Clearly, this is a particular activation of $\Gamma, n_2 : T_1 \prec T_2 \vdash_G \Delta$, and any other activation implies $\mathsf{D}(G), n_1 \preccurlyeq_+ n_2; \mathsf{D}(\Gamma)^+, \mathsf{D}(\Delta)^- \vdash -\mathsf{D}(T_2) \rightarrow_- \mathsf{D}(T_1) \mathbin{@} n_2$ is derivable by the left-to-right lemma, and hence are derivable.

Case.

$$\frac{\begin{array}{l} n_1 G n_2 \\ \Gamma \vdash_G n_1 : T_1, \Delta \\ \Gamma, n_1 : T_2 \vdash_G \Delta \end{array}}{\Gamma \vdash_G n_2 : T_1 \prec T_2, \Delta}\ \textsc{subR}$$

This case follows in the same way as the case for IMPL, except the particular activation of $\Gamma, n_1 : T_2 \vdash_G \Delta$ has to have the active formulas such that the rule IMPBAR can be applied.

A.6. **Proof of Lemma 36.** This is a proof by induction on the assumed typing derivation.

Case.

$$\frac{G \vdash n \preccurlyeq_p^* n'}{G; \Gamma, p\ A \mathbin{@} n, \Gamma' \vdash p\ A \mathbin{@} n'}\ \textsc{ax}$$

We only show the case when $p = +$, because the case when $p = -$ is similar. By the definition of the L-translation we must show that the L-sequent $\mathsf{L}(\Gamma)^+, n : \mathsf{L}(A), \mathsf{L}(\Gamma')^+ \vdash_{\mathsf{L}(G)} \mathsf{L}(\Gamma)^-, n' : \mathsf{L}(A), \mathsf{L}(\Gamma')^-$ is derivable. Since we know that for any $n_1, n_2 \in |n \preccurlyeq_{p'} n, G|, |\Gamma|$, if $G \vdash n_1 \preccurlyeq_{p'}^* n_2$, then $n_1 \preccurlyeq_{p'} n_2 \in G$, it must be the case that $n \preccurlyeq_+ n' \in G$, and thus, $n\mathsf{L}(G)n'$. At this point we may apply the L inference rule MONL using this fact. Therefore, we have derived $\mathsf{L}(\Gamma)^+, n : \mathsf{L}(A), n' : \mathsf{L}(A), \mathsf{L}(\Gamma')^+ \vdash_{\mathsf{L}(G)} \mathsf{L}(\Gamma)^-, n' : \mathsf{L}(A), \mathsf{L}(\Gamma')^-$, and then we may complete the derivation by applying the L inference rule HYP.

Case.

$$\frac{}{G; \Gamma \vdash p\ \langle p \rangle \mathbin{@} n}\ \textsc{unit}$$

Suppose $p = +$. Then by the definition of the L-translation we must derive $\mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n : \top, \mathsf{L}(\Gamma)^-$, but this follows by simply applying the L inference rule TRUER.

Suppose $p = -$. Then by the definition of the L-translation we must derive $\mathsf{L}(\Gamma)^+, n : \perp \vdash_{\mathsf{L}(G)} \mathsf{L}(\Gamma)^-$, but this follows by simply applying the L inference rule FALSEL.

Case.

$$\frac{G; \Gamma \vdash p\ A \mathbin{@} n \qquad G; \Gamma \vdash p\ B \mathbin{@} n}{G; \Gamma \vdash p\ (A \wedge_p B) \mathbin{@} n}\ \textsc{and}$$

Suppose $p = +$. Then by the induction hypothesis we know the following:

$$\mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n : A, \mathsf{L}(\Gamma)^-$$
$$\mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n : B, \mathsf{L}(\Gamma)^-$$

By the definition of the L-translation we must show that

$$\mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n : A \wedge B, \mathsf{L}(\Gamma)^-.$$

This easily follows by applying the L inference rule andR.

Now suppose $p = -$. Then by the induction hypothesis we know the following:

$$\mathsf{L}(\Gamma)^+, n : A \vdash_{\mathsf{L}(G)} \mathsf{L}(\Gamma)^-$$
$$\mathsf{L}(\Gamma)^+, n : B \vdash_{\mathsf{L}(G)} \mathsf{L}(\Gamma)^-$$

By the definition of the L-translation we must show that

$$\mathsf{L}(\Gamma)^+, n : A \vee B \vdash_{\mathsf{L}(G)} \mathsf{L}(\Gamma)^-.$$

This easily follows by applying the L inference rule disjL.

Case.

$$\frac{G; \Gamma \vdash p\ A_d \ @\ n}{G; \Gamma \vdash p\ (A_1 \wedge_{\bar{p}} A_2)\ @\ n}\ \textsc{andBar}$$

Suppose $p = +$ and $d = 1$. Then by the induction hypothesis we know the following:

$$\mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n : \mathsf{L}(A_1), \mathsf{L}(\Gamma)^-$$

Then by the L admissible inference rule weakR (Lemma 33) we know the following:

$$\mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n : \mathsf{L}(A_1), n : \mathsf{L}(A_2), \mathsf{L}(\Gamma)^-$$

Thus, we obtain our result that $\mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n : \mathsf{L}(A_1) \vee \mathsf{L}(A_2), \mathsf{L}(\Gamma)^-$ is derivable by applying the L inference rule disjR. The case for when $d = 2$ is similar.

If $p = -$ then the result follows similarly to the case when $p = +$ except that the derivation is a result of applying the rule andL after applying the admissible L inference rule weakL to the induction hypothesis.

Case.

$$\frac{\begin{array}{l} n' \notin |G|, |\Gamma| \\ (G, n \preccurlyeq_p n'); \Gamma, p\ A \ @\ n' \vdash p\ B \ @\ n' \end{array}}{G; \Gamma \vdash p\ (A \rightarrow_p B)\ @\ n}\ \textsc{imp}$$

Suppose $p = +$. Then by the induction hypothesis we know the following:

$$\mathsf{L}(\Gamma)^+, n' : \mathsf{L}(A) \vdash_{(\mathsf{L}(G) \cup \{(n,n')\})} n' : \mathsf{L}(B), \mathsf{L}(\Gamma)^-$$

We know by assumption that $n' \notin |G|, |\Gamma|$, and hence, $n' \notin |\mathsf{L}(G)|, |\mathsf{L}(\Gamma)^+|, |\mathsf{L}(\Gamma)^-|$ by the definition of the L translation. Therefore, our result follows by simply applying the L inference rule impR.

Suppose $p = -$. This case follows similarly to the case when $p = +$, but we conclude with the L inference rule subL.

Case.

$$\frac{\begin{array}{l} G \vdash n \preccurlyeq^*_{\bar{p}} n' \\ G;\Gamma \vdash \bar{p}\ A @ n' \qquad G;\Gamma \vdash p\ B @ n' \end{array}}{G;\Gamma \vdash p\ (A \rightarrow_{\bar{p}} B) @ n}\ \textsc{impBar}$$

Suppose $p = +$. Then by the induction hypothesis we know the following:

$$\begin{array}{rl} i. & \mathsf{L}(\Gamma)^+, n' : \mathsf{L}(A) \vdash_{\mathsf{L}(G)} \mathsf{L}(\Gamma)^- \\ ii. & \mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n' : \mathsf{L}(B), \mathsf{L}(\Gamma)^- \end{array}$$

Furthermore, we know for any $n_1, n_2 \in |n \preccurlyeq_{p'} n, G|, |\Gamma|$ if $G \vdash n_1 \preccurlyeq^*_{p'} n_2$, then $n_1 \preccurlyeq_{p'} n_2 \in G$, and we know by assumption that $G \vdash n \preccurlyeq^*_{-} n'$, and thus, $n \preccurlyeq_{-} n' \in G$, hence, $n' \mathsf{L}(G) n$ by the definition of the L-translation.

It suffices to show that $\mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n : \mathsf{L}(B) \prec \mathsf{L}(A), \mathsf{L}(\Gamma)^-$, but this follows by applying the L inference rule subR using i and ii from above as well as the fact that we know $n' \mathsf{L}(G) n$.

Now suppose $p = -$. Similar to the case when $p = +$, but we conclude with applying the L inference rule impL, and the induction hypothesis provides the following:

$$\begin{array}{rl} i. & \mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n' : \mathsf{L}(A), \mathsf{L}(\Gamma)^- \\ ii. & \mathsf{L}(\Gamma)^+, n' : \mathsf{L}(B) \vdash_{\mathsf{L}(G)} \mathsf{L}(\Gamma)^-. \end{array}$$

Case.

$$\frac{p\ B @ n' \in (\Gamma, \bar{p}\ A @ n) \qquad G;\Gamma, \bar{p}\ A @ n \vdash \bar{p}\ B @ n'}{G;\Gamma \vdash p\ A @ n}\ \textsc{axCut}$$

Suppose $p = +$. Then by the induction hypothesis we know the following:

$$\mathsf{L}(\Gamma)^+, n' : \mathsf{L}(B) \vdash_{\mathsf{L}(G)} n : \mathsf{L}(A), \mathsf{L}(\Gamma)^-$$

Now we know that $p\ B @ n' \in (\Gamma, \bar{p}\ A @ n)$, and hence, $n' : \mathsf{L}(B) \in \mathsf{L}(\Gamma)^+$ which implies we know the following:

$$\mathsf{L}(\Gamma_1)^+, n' : \mathsf{L}(B), \mathsf{L}(\Gamma_2)^+, n' : \mathsf{L}(B) \vdash_{\mathsf{L}(G)} n : \mathsf{L}(A), \mathsf{L}(\Gamma)^-$$

Therefore, by applying the admissible L inference rule contrL we know the following:

$$\mathsf{L}(\Gamma_1)^+, n' : \mathsf{L}(B), \mathsf{L}(\Gamma_2)^+ \vdash_{\mathsf{L}(G)} n : \mathsf{L}(A), \mathsf{L}(\Gamma)^-$$

This is equivalent to our result:

$$\mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n : \mathsf{L}(A), \mathsf{L}(\Gamma)^-$$

Suppose $p = -$. Then by the induction hypothesis we know the following:

$$\mathsf{L}(\Gamma)^+, n' : \mathsf{L}(A) \vdash_{\mathsf{L}(G)} n : \mathsf{L}(B), \mathsf{L}(\Gamma)^-$$

This case now follows similarly to the previous case by exposing $n : \mathsf{L}(B)$ in $\mathsf{L}(\Gamma)^-$, and then using contraction on the right.

Case.

$$\frac{\bar{p}\ B @ n' \in (\Gamma, \bar{p}\ A @ n) \qquad G;\Gamma, \bar{p}\ A @ n \vdash p\ B @ n'}{G;\Gamma \vdash p\ A @ n}\ \textsc{axCutBar}$$

This case is similar to the previous case except in the case when $p = +$ we use contraction on the right, and then when $p = -$ we use contraction on the left.

A.7. **Proof of Lemma 42: DIL-validity is L-validity.** Suppose $[\![G; \Gamma \vdash p\, A @ n]\!]_N$ holds for some Kripke model $(W, R, V)$ and node interpreter $N$ on $|G|$, and $p = +$. It suffices to show that $\mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n : \mathsf{L}(A), \mathsf{L}(\Gamma)^-$ is L-valid. By the definition of the interpretation of DIL-sequents (Definition 11) we know that

$$\text{if } [\![G]\!]_N \text{ and } [\![\Gamma]\!]_N, \text{ then } p[\![A]\!]_{(N\,n)}$$

Now to show that $\mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n : \mathsf{L}(A), \mathsf{L}(\Gamma)^-$ is L-valid we must show that at least one of the following does not hold:

i. for any $n_1 \mathsf{L}(G) n_2$, $R\,(N\,n_1)\,(N\,n_2)$
ii. for any $n : \mathsf{L}(B) \in \mathsf{L}(\Gamma)^+$, $[\![\mathsf{L}(B)]\!]_{N\,n}$
iii. for any $n : \mathsf{L}(B) \in (n : \mathsf{L}(A), \mathsf{L}(\Gamma)^-)$, $\neg[\![\mathsf{L}(B)]\!]_{N\,n}$

So if neither $[\![G]\!]_N$ or $[\![\Gamma]\!]_N$ hold, then neither of i or ii will hold. Thus, $\mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n : \mathsf{L}(A), \mathsf{L}(\Gamma)^-$ is L-valid.

So assume $[\![G]\!]_N$ or $[\![\Gamma]\!]_N$ hold. Then both i and ii are satisfied by Lemma 40. However, we now know $+[\![A]\!]_{(N\,n)} = [\![A]\!]_{(N\,n)}$ holds, and hence by Lemma 40, iii does not hold. Therefore, $\mathsf{L}(\Gamma)^+ \vdash_{\mathsf{L}(G)} n : \mathsf{L}(A), \mathsf{L}(\Gamma)^-$ is L-valid.

Now suppose $p = -$. It suffices to show that $\mathsf{L}(\Gamma)^+, n : \mathsf{L}(A) \vdash_{\mathsf{L}(G)} \mathsf{L}(\Gamma)^-$ is L-valid. However, notice that we must show that at least one of the following does not hold:

i. for any $n_1 \mathsf{L}(G) n_2$, $R\,(N\,n_1)\,(N\,n_2)$
ii. for any $n : \mathsf{L}(B) \in (\mathsf{L}(\Gamma)^+, n : \mathsf{L}(A))$, $[\![\mathsf{L}(B)]\!]_{N\,n}$
iii. for any $n : \mathsf{L}(B) \in \mathsf{L}(\Gamma)^-$, $\neg[\![\mathsf{L}(B)]\!]_{N\,n}$

However, notice that ii will allows be false in this case, because if $[\![G]\!]_N$ or $[\![\Gamma]\!]_N$ hold, then we know $-[\![A]\!]_{(N\,n)} = \neg[\![A]\!]_{(N\,n)}$, which implies that $\neg[\![\mathsf{L}(A)]\!]_{(N\,n)}$. Therefore, $\mathsf{L}(\Gamma)^+, n : \mathsf{L}(A) \vdash_{\mathsf{L}(G)} \mathsf{L}(\Gamma)^-$ is L-valid.

## Appendix B. Proofs from Section 5: Dualized Type Theory

B.1. **Proof of Lemma 44.** Due to the size of the derivation in question we give several derivations which compose together to form the typing derivation of $G; \Gamma \vdash \mathbf{case}\; t\; \mathbf{of}\; x.t_1, x.t_2 : p\; C @ n$.

The typing derivation begins using cut as follows:

$$\frac{D_0 \qquad D_1}{G; \Gamma \vdash \nu\, z_0.(\nu\, z_1.(\nu\, z_2.t \bullet (z_1, z_2)) \bullet (\nu\, x.t_2 \bullet z_0)) \bullet (\nu\, x.t_1 \bullet z_0) : +\, C @ n}\ \textsc{cut}$$

Then the remainder of the derivation depends on the following sub-derivations:

$D_0$ :

$$\frac{D_3 \qquad D_4}{G; \Gamma, z_0 : -\, C @ n \vdash \nu\, z_1.(\nu\, z_2.t \bullet (z_1, z_2)) \bullet (\nu\, x.t_2 \bullet z_0) : +\, A @ n}\ \textsc{cut}$$

$D_1$ :

$$\dfrac{D_2 \qquad \dfrac{}{G; \Gamma, z_0 : -\, C \mathbin{@} n, x : +\, A \mathbin{@} n \vdash z_0 : -\, C \mathbin{@} n}\ \textsc{ax}}{G; \Gamma, z_0 : -\, C \mathbin{@} n \vdash \nu\, x.t_1 \bullet z_0 : -\, A \mathbin{@} n}\ \textsc{cut}$$

$D_2$ :

$$\dfrac{G; \Gamma, x : +\, A \mathbin{@} n \vdash t_1 : +\, C \mathbin{@} n}{G; \Gamma, z_0 : -\, C \mathbin{@} n, x : +\, A \mathbin{@} n \vdash t_1 : +\, C \mathbin{@} n}\ \textsc{weakening}$$

$D_4$ :

$$\dfrac{D_5 \qquad G; \Gamma, z_0 : -\, C \mathbin{@} n, z_1 : -\, A \mathbin{@} n, x : +\, B \mathbin{@} n \vdash z_0 : -\, C \mathbin{@} n}{G; \Gamma, z_0 : -\, C \mathbin{@} n, z_1 : -\, A \mathbin{@} n \vdash \nu\, x.t_2 \bullet z_0 : -\, B \mathbin{@} n}\ \textsc{cut}$$

$D_3$ :

$$\dfrac{D_6 \qquad D_7}{G; \Gamma, z_0 : -\, C \mathbin{@} n, z_1 : -\, A \mathbin{@} n \vdash \nu\, z_2.t \bullet (z_1, z_2) : +\, B \mathbin{@} n}\ \textsc{cut}$$

$D_5$ :

$$\dfrac{G; \Gamma, x : +\, B \mathbin{@} n \vdash t_2 : +\, C \mathbin{@} n}{G; \Gamma, z_0 : -\, C \mathbin{@} n, z_1 : -\, A \mathbin{@} n, x : +\, B \mathbin{@} n \vdash t_2 : +\, C \mathbin{@} n}\ \textsc{weakening}$$

$D_6$ :

$$\dfrac{G; \Gamma \vdash t : +\, (A \wedge_- B) \mathbin{@} n}{G; \Gamma, z_0 : -\, C \mathbin{@} n, z_1 : -\, A \mathbin{@} n, z_2 : -\, B \mathbin{@} n \vdash t : +\, (A \wedge_- B) \mathbin{@} n}\ \textsc{weakening}$$

$D_7$ :

$$\dfrac{D_8 \qquad D_9}{G; \Gamma, z_0 : -\, C \mathbin{@} n, z_1 : -\, A \mathbin{@} n, z_2 : -\, B \mathbin{@} n \vdash (z_1, z_2) : -\, (A \wedge_- B) \mathbin{@} n}\ \textsc{and}$$

$D_8$ :

$$\dfrac{}{G; \Gamma, z_0 : -\, C \mathbin{@} n, z_1 : -\, A \mathbin{@} n, z_2 : -\, B \mathbin{@} n \vdash z_1 : -\, A \mathbin{@} n}\ \textsc{ax}$$

$D_9$ :

$$\dfrac{}{G; \Gamma, z_0 : -\, C \mathbin{@} n, z_1 : -\, A \mathbin{@} n, z_2 : -\, B \mathbin{@} n \vdash z_2 : -\, B \mathbin{@} n}\ \textsc{ax}$$

## Appendix C. Proofs from Section 6: Metatheory of DTT

C.1. **Proof of Lemma 46: Node Renaming.** This is a proof by induction on the assumed reachability derivation. Throughout each case suppose we have nodes $n_4$ and $n_5$.

Case.

$$\dfrac{}{G, n_1 \preccurlyeq_p n_3, G' \vdash n_1 \preccurlyeq_p^* n_3}\ \textsc{ax}$$

Trivial.

Case.

$$\dfrac{}{G_1, G_2 \vdash n \preccurlyeq_p^* n}\ \textsc{refl}$$

Trivial.

Case.

$$\frac{G_1, G_2 \vdash n_1 \preccurlyeq^*_p n' \qquad G_1, G_2 \vdash n' \preccurlyeq^*_p n_3}{G_1, G_2 \vdash n_1 \preccurlyeq^*_p n_3}\ \textsc{trans}$$

By the induction hypothesis we know that for any nodes $n_4'$ and $n_5'$ we have $[n_4'/n_5']G_1, [n_4'/n_5']G_2 \vdash [n_4'/n_5']n_1 \preccurlyeq^*_p [n_4'/n_5']n'$, and for any nodes $n_4''$ and $n_5''$ we have $[n_4''/n_5'']G_1, [n_4''/n_5'']G_2 \vdash [n_4''/n_5'']n' \preccurlyeq^*_p [n_4''/n_5'']n_3$. Choose $n_4$ for $n_4'$ and $n_4''$ and $n_5$ for $n_5'$ and $n_5''$ to obtain $[n_4/n_5]G_1, [n_4/n_5]G_2 \vdash [n_4/n_5]n_1 \preccurlyeq^*_p [n_4/n_5]n'$ and $[n_4/n_5]G_1, [n_4/n_5]G_2 \vdash [n_4/n_5]n' \preccurlyeq^*_p [n_4/n_5]n_3$. Finally, this case follows by reapplying the rule to the previous two facts.

Case.

$$\frac{G \vdash n' \preccurlyeq^*_{\bar{p}} n}{G \vdash n \preccurlyeq^*_p n'}\ \textsc{flip}$$

Similar to the previous case.

C.2. **Proof of Lemma 47: Node Substitution for Reachability.** This is a proof by induction on the form of the assumed reachability derivation. Throughout the following cases we assume $G, G' \vdash n_1 \preccurlyeq^*_{p_1} n_3$ holds.

Case.

$$\frac{}{G_1, n_4 \preccurlyeq_p n_5, G_2 \vdash n_4 \preccurlyeq^*_p n_5}\ \textsc{ax}$$

Suppose $G_1, n_4 \preccurlyeq_p n_5, G_2 = G, n_1 \preccurlyeq_{p_1} n_2, G'$. Then either $n_1 \preccurlyeq_{p_1} n_2 \in G_1$, $n_1 \preccurlyeq_{p_1} n_2 \in G_2$, or $n_1 \preccurlyeq_{p_1} n_2 \equiv n_4 \preccurlyeq_p n_5$. Suppose $n_1 \preccurlyeq_{p_1} n_2 \in G_1$, then $G_1 = G_1', n_1 \preccurlyeq_p n_2, G_1''$. Then it is easy to see that $[n_3/n_2]G_1', [n_3/n_2]G_1'', [n_3/n_2]n_4 \preccurlyeq_p [n_3/n_2]n_5, [n_3/n_2]G_2 \vdash [n_3/n_2]n_4 \preccurlyeq^*_p [n_3/n_2]n_5$ is derivable by applying Ax. The case where $n_1 \preccurlyeq_{p_1} n_2 \in G_2$ is similar.

Now suppose $n_1 \preccurlyeq_{p_1} n_2 \equiv n_4 \preccurlyeq_p n_5$. Then we know by assumption that

$$\frac{}{G_1, n_1 \preccurlyeq_p n_2, G_2 \vdash n_1 \preccurlyeq^*_p n_2}\ \textsc{ax}$$

Then it suffices to show $[n_3/n_2]G_1, [n_3/n_2]G_2 \vdash [n_3/n_2]n_1 \preccurlyeq^*_p [n_3/n_2]n_2$, which is equivalent to $[n_3/n_2]G_1, [n_3/n_2]G_2 \vdash [n_3/n_2]n_1 \preccurlyeq^*_p n_3$. Now if $n_1$ is equivalent to $n_2$, then $[n_3/n_2]G_1, [n_3/n_2]G_2 \vdash [n_3/n_2]n_1 \preccurlyeq^*_p n_3$ holds by reflexivity, and if $n_1$ is distinct from $n_2$, then $[n_3/n_2]G_1, [n_3/n_2]G_2 \vdash [n_3/n_2]n_1 \preccurlyeq^*_p n_3$ is equivalent to $[n_3/n_2]G_1, [n_3/n_2]G_2 \vdash n_1 \preccurlyeq^*_p n_3$. We know by assumption that $G, G' \vdash n_1 \preccurlyeq^*_{p_1} n_3$ holds, which is equivalent to $G_1, G_2 \vdash n_1 \preccurlyeq^*_p n_3$. Now if $n_3$ is equal to $n_2$, then $[n_3/n_2]G_1, [n_3/n_2]G_2 \vdash n_1 \preccurlyeq^*_p n_3$ is equivalent to $G_1, G_2 \vdash n_1 \preccurlyeq^*_p n_3$. So suppose $n_3$ is distinct from $n_2$, then by Lemma 46 we know $[n_3/n_2]G_1, [n_3/n_2]G_2 \vdash n_1 \preccurlyeq^*_p n_3$.

Case.

$$\frac{}{G, n_1 \preccurlyeq_{p_1} n_2, G' \vdash n \preccurlyeq^*_p n}\ \textsc{refl}$$

Trivial.

Case.

$$\frac{G, n_1 \preccurlyeq_{p_1} n_2, G' \vdash n_4 \preccurlyeq^*_p n_6 \qquad G \vdash n_6 \preccurlyeq^*_p n_5}{G, n_1 \preccurlyeq_{p_1} n_2, G' \vdash n_4 \preccurlyeq^*_p n_5}\ \textsc{trans}$$

This case by applying the induction to each premise, and then reapplying the rule.

Case.

$$\frac{G, n_1 \preccurlyeq_{p_1} n_2, G' \vdash n_5 \preccurlyeq^*_{\bar{p}} n_4}{G, n_1 \preccurlyeq_{p_1} n_2, G' \vdash n_4 \preccurlyeq^*_p n_5}\ \textsc{flip}$$

This case holds by applying the induction hypothesis to the premise, and then reapplying the rule.

### C.3. **Proof of Lemma 48: Node Substitution for Typing.**

This is a proof by induction on the form of the assumed typing derivation. Throughout each of the following cases we assume $G, G' \vdash n_1 \preccurlyeq^*_{p_1} n_4$ holds.

Case.

$$\frac{G, n_1 \preccurlyeq_{p_1} n_2, G' \vdash n \preccurlyeq^*_p n_3}{G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma_1, y : p_2\ A \,@\, n, \Gamma_2 \vdash y : p_2\ A \,@\, n_3}\ \textsc{Ax}$$

First, by node substitution for reachability (Lemma 47) we know $[n_4/n_2]G, [n_4/n_2]G' \vdash [n_4/n_2]n \preccurlyeq^*_p [n_4/n_2]n_3$. Thus, by applying the Ax rule we may derive $[n_4/n_2]G, [n_4/n_2]G'; [n_4/n_2]\Gamma_1, y : p_2\ A \,@\, [n_4/n_2]n, [n_4/n_2]\Gamma_2 \vdash y : p_2\ A \,@\, [n_4/n_2]n_3$.

Case.

$$\frac{}{G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma \vdash \mathbf{triv} : p_2\ \langle p_2 \rangle \,@\, n_3}\ \textsc{Unit}$$

Trivial.

Case.

$$\frac{G, n_1 \preccurlyeq_{p_1} n_2; \Gamma \vdash t_1 : p_2\ A_1 \,@\, n_3 \qquad G, n_1 \preccurlyeq_{p_1} n_2; \Gamma \vdash t_2 : p_2\ A_2 \,@\, n_3}{G, n_1 \preccurlyeq_{p_1} n_2; \Gamma \vdash (t_1, t_2) : p_2\ (A_1 \wedge_{p_2} A_2) \,@\, n_3}\ \textsc{And}$$

This case holds by applying the induction hypothesis to each premise, and then reapplying the rule.

Case.

$$\frac{G, n_1 \preccurlyeq_{p_1} n_2; \Gamma \vdash t' : p_2\ A_d \,@\, n_3}{G, n_1 \preccurlyeq_{p_1} n_2; \Gamma \vdash \mathbf{in}_d\ t' : p_2\ (A_1 \wedge_{\bar{p}_2} A_2) \,@\, n_3}\ \textsc{AndBar}$$

This case holds by applying the induction hypothesis to the premise, and then reapplying the rule.

Case.

$$\frac{\begin{array}{l} n' \notin |G, n_1 \preccurlyeq_{p_1} n_2, G'|, |\Gamma| \\ (G, n_1 \preccurlyeq_{p_1} n_2, G', n_3 \preccurlyeq_{p_2} n'); \Gamma, x : p_2\, A_1 \,@\, n' \vdash t' : p_2\, A_2 \,@\, n' \end{array}}{G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma \vdash \lambda x.t' : p_2\, (A_1 \to_{p_2} A_2) \,@\, n_3}\ \textsc{Imp}$$

First, if $n' \notin |G, n_1 \preccurlyeq_{p_1} n_2, G'|, |\Gamma|$, then $n' \notin |G, G'|, |\Gamma|$. Furthermore, we know that $[n_4/n_2]n' \notin |[n_4/n_2]G, [n_4/n_2]G'|, |[n_4/n_2]\Gamma|$, because we know $n'$ is distinct from $n_2$ by assumption, and if $n'$ is equal to $n_4$, then $n' \notin |G, n_1 \preccurlyeq_{p_1} n_2, G'|, |\Gamma|$ implies that $n_1$ must also be $n_4$, because we know by assumption that $G, G' \vdash n_1 \preccurlyeq^*_{p_1} n_4$ which could only be derived by reflexivity since $n' \notin |G, G'|, |\Gamma|$, but we know by assumption that $n' \notin |G, n_1 \preccurlyeq_{p_1} n_2, G'|, |\Gamma|$ which implies that $n'$ must be distinct from $n_1$, and hence a contradiction, thus $n'$ cannot be $n_4$. Therefore, we know $n' \notin |[n_4/n_2]G, [n_4/n_2]G'|, |[n_4/n_2]\Gamma|$.

By the induction hypothesis we know

$$([n_4/n_2]G, [n_4/n_2]G', [n_4/n_2]n_3 \preccurlyeq_{p_2} [n_4/n_2]n'); [n_4/n_2]\Gamma, x : p_2\, A_1 \,@\, [n_4/n_2]n' \vdash t' : p_2\, A_2 \,@\, [n_4/n_2]n'$$

which is equivalent to

$$([n_4/n_2]G, [n_4/n_2]G', [n_4/n_2]n_3 \preccurlyeq_{p_2} n'); [n_4/n_2]\Gamma, x : p_2\, A_1 \,@\, n' \vdash t' : p_2\, A_2 \,@\, n'.$$

Finally, this case follows by applying the Imp rule using $n' \notin |[n_4/n_2]G, [n_4/n_2]G'|, |[n_4/n_2]\Gamma|$ and the previous fact.

Case.

$$\frac{\begin{array}{l} G, n_1 \preccurlyeq_{p_1} n_2, G' \vdash n_3 \preccurlyeq^*_{\bar{p_2}} n' \\ G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma \vdash t_1 : \bar{p_2}\, A_1 \,@\, n' \qquad G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma \vdash t_2 : p_2\, A_2 \,@\, n' \end{array}}{G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma \vdash \langle t_1, t_2 \rangle : p_2\, (A_1 \to_{\bar{p_2}} A_2) \,@\, n_3}\ \textsc{ImpBar}$$

We now by assumption that $G, G' \vdash n_1 \preccurlyeq^*_{p_1} n_4$ holds. So by node substitution for reachability (Lemma 47) we know $[n_4/n_2]G, [n_4/n_2]G' \vdash [n_4/n_2]n_3 \preccurlyeq^*_{\bar{p_2}} [n_4/n_2]n'$. Now by the induction hypothesis we know $[n_4/n_2]G, [n_4/n_2]G'; [n_4/n_2]\Gamma \vdash t_1 : \bar{p_2}\, A_1 \,@\, [n_4/n_2]n'$ and
$[n_4/n_2]G, [n_4/n_2]G'; [n_4/n_2]\Gamma \vdash t_2 : p_2\, A_2 \,@\, [n_4/n_2]n'$. This case then follows by applying the rule ImBar to the previous three facts.

Case.

$$\frac{\begin{array}{l} G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma, y : \bar{p_2}\, A \,@\, n_3 \vdash t_1 : +\, C \,@\, n \\ G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma, y : \bar{p_2}\, A \,@\, n_3 \vdash t_2 : -\, C \,@\, n \end{array}}{G, n_1 \preccurlyeq_{p_1} n_2, G'; \Gamma \vdash \nu\, x.t_1 \bullet t_2 : p_2\, A \,@\, n_3}\ \textsc{Cut}$$

This case follows by applying the induction hypothesis to each premise, and then reapplying the rule.

C.4. **Proof of Lemma 49: Substitution for Typing.** This proof holds by a straightforward induction on the second assumed typing relation.

Case.

$$\frac{G \vdash n \preccurlyeq^*_{p} n'}{G; \Gamma_1, y : p\, C \,@\, n, \Gamma_2 \vdash y : p\, C \,@\, n'}\ \textsc{Ax}$$

Trivial.

Case.

$$\frac{}{G; \Gamma_1 \vdash \mathbf{triv} : p \, \langle p \rangle \, @ \, n} \; \text{UNIT}$$

Trivial.

Case.

$$\frac{G; \Gamma_1 \vdash t_1' : p \, A \, @ \, n \qquad G; \Gamma_1 \vdash t_2' : p \, B \, @ \, n}{G; \Gamma_1 \vdash (t_1', t_2') : p \, (C_1 \wedge_p C_2) \, @ \, n} \; \text{AND}$$

Suppose $\Gamma_1 \equiv \Gamma, x : p_1 \, A \, @ \, n_1, \Gamma'$. Then this case follows from applying the induction hypothesis to each premise and then reapplying the rule.

Case.

$$\frac{G; \Gamma_1 \vdash t : p \, C_d \, @ \, n}{G; \Gamma_1 \vdash \mathbf{in}_d \, t : p \, (C_1 \wedge_{\bar{p}} C_2) \, @ \, n} \; \text{ANDBAR}$$

Suppose $\Gamma_1 \equiv \Gamma, x : p_1 \, A \, @ \, n_1, \Gamma'$. Then this case follows from applying the induction hypothesis to the premise and then reapplying the rule.

Case.

$$\frac{\begin{array}{l} n' \notin |G|, |\Gamma_1| \\ (G, n \preccurlyeq_p n'); \Gamma_1, x : p \, C_1 \, @ \, n' \vdash t : p \, C_2 \, @ \, n' \end{array}}{G; \Gamma_1 \vdash \lambda x.t : p \, (C_1 \rightarrow_p C_2) \, @ \, n} \; \text{IMP}$$

Similarly to the previous case.

Case.

$$\frac{\begin{array}{l} G \vdash n \preccurlyeq_{\bar{p}}^{*} n' \\ G; \Gamma_1 \vdash t_1' : \bar{p} \, C_1 \, @ \, n' \qquad G; \Gamma_1 \vdash t_2' : p \, C_2 \, @ \, n' \end{array}}{G; \Gamma_1 \vdash \langle t_1', t_2' \rangle : p \, (C_1 \rightarrow_{\bar{p}} C_2) \, @ \, n} \; \text{IMPBAR}$$

Suppose $\Gamma_1 \equiv \Gamma, x : p_1 \, A \, @ \, n_1, \Gamma'$. Then this case follows from applying the induction hypothesis to each premise and then reapplying the rule.

Case.

$$\frac{\begin{array}{c} G; \Gamma_1, y : \bar{p} \, C \, @ \, n \vdash t_1' : + \, C' \, @ \, n' \\ G; \Gamma_1, y : \bar{p} \, C \, @ \, n \vdash t_2' : - \, C' \, @ \, n' \end{array}}{G; \Gamma_1 \vdash \nu \, x.t_1' \bullet t_2' : p \, C \, @ \, n} \; \text{CUT}$$

Similarly to the previous case.

C.5. **Proof of Lemma 50: Type Preservation.** This is a proof by induction on the form of the assumed typing derivation. We only consider non-trivial cases. All the other cases either follow directly from assumptions or are similar to the cases we provide below.

Case.

$$\frac{\begin{array}{c} G;\Gamma, x : \bar{p}\ A \mathbin{@} n \vdash t_1 : +\ B \mathbin{@} n' \\ G;\Gamma, x : \bar{p}\ A \mathbin{@} n \vdash t_2 : -\ B \mathbin{@} n' \end{array}}{G;\Gamma \vdash \nu\, x.t_1 \bullet t_2 : p\ A \mathbin{@} n}\ \textsc{Cut}$$

The interesting cases are the ones where the assumed cut is a redex itself, otherwise this case holds by the induction hypothesis. Thus, we case split on the form of this redex.

Case. Suppose $\nu\, x.t_1 \bullet t_2 \equiv \nu\, x.\lambda y.t'_1 \bullet \langle t'_2, t''_2\rangle$, thus, $t_1 \equiv \lambda y.t'_1$ and $t_2 \equiv \langle t'_2, t''_2\rangle$. This then implies that $B \equiv B_1 \to_+ B_2$ for some $B_1$ and $B_2$. Then

$$t \equiv \nu\, x.t_1 \bullet t_2 \equiv \nu\, x.\lambda y.t'_1 \bullet \langle t'_2, t''_2\rangle \rightsquigarrow \nu\, x.[t'_2/y]t'_1 \bullet t''_2 \equiv t'.$$

Now by inversion we know the following:

(1) $G, (n' \preccurlyeq_+ n''); \Gamma, x : \bar{p}\ A \mathbin{@} n, y : +\ B_1 \mathbin{@} n'' \vdash t'_1 : +\ B_2 \mathbin{@} n''$
for some $n'' \notin |G|, |\Gamma, x : \bar{p}\ A \mathbin{@} n|$

(2) $G;\Gamma, x : \bar{p}\ A \mathbin{@} n \vdash t'_2 : +\ B_1 \mathbin{@} n'''$

(3) $G;\Gamma, x : \bar{p}\ A \mathbin{@} n \vdash t''_2 : -\ B_2 \mathbin{@} n'''$

(4) $G \vdash n' \preccurlyeq^*_+ n'''$

Using (1) and (4) we may apply node substitution for typing (Lemma 48) to obtain

$$(5)\ [n'''/n'']G; [n'''/n'']\Gamma, x : \bar{p}\ A \mathbin{@} n, y : +\ B_1 \mathbin{@} n''' \vdash t'_1 : +\ B_2 \mathbin{@} n'''.$$

Finally, by applying substitution for typing using (2) and (5) we obtain

$$(6)\ [n'''/n'']G; [n'''/n'']\Gamma, x : \bar{p}\ A \mathbin{@} n \vdash [t'_2/y]t'_1 : +\ B_2 \mathbin{@} n''',$$

and since $n''$ is a fresh in $G$ and $\Gamma$ we know (6) is equivalent to

$$(7)\ G;\Gamma, x : \bar{p}\ A \mathbin{@} n \vdash [t'_2/y]t'_1 : +\ B_2 \mathbin{@} n'''.$$

Finally, by applying the Cut rule using (7) and (3) we obtain

$$G;\Gamma \vdash \nu\, x.[t'_2/y]t'_1 \bullet t''_2 : p\ A \mathbin{@} n.$$

C.6. **Proof of Lemma 53: SN Interpretations.** For purposes of this proof and subsequent ones, define $\delta(t)$ to be the length of the longest reduction sequence from $t$ to a normal form, for $t \in \mathbf{SN}$.

The proof of the lemma is by mutual well-founded induction on the pair $(A, n)$, where $n$ is the number of the proposition in the statement of the lemma; the well-founded ordering in question is the lexicographic combination of the structural ordering on types (for $A$) and the ordering $1 > 2 > 4 > 3$ (for $n$).

For proposition (1): assume $t \in [\![A]\!]^+$, and show $t \in \mathbf{SN}$. Let $x$ be a variable. By IH(2), $x \in [\![A]\!]^-$, so by the definition of $[\![A]\!]^+$, we have

$$\nu\, x.t \bullet x \in \mathbf{SN}$$

This implies $t \in \mathbf{SN}$.

For proposition (2): assume $x \in \mathit{Vars}$, and show $x \in [\![A]\!]^-$. For the latter, it suffices to assume arbitrary $y \in \mathit{Vars}$ and $t' \in [\![A]\!]^{+c}$, and show $\nu\, y.t' \bullet x \in \mathbf{SN}$. We will prove this by inner induction on $\delta(t')$, which is defined by IH(4). By the definition of $[\![A]\!]^{+c}$ for the various cases of $A$, we see that $\nu\, y.t' \bullet x$ cannot be a redex itself, as $t'$ cannot be a cut. If $t'$ is a normal form we are done. If $t \rightsquigarrow t''$, then we have $t'' \in [\![A]\!]^{+c}$ by Lemma 52, and we may apply the inner induction hypothesis.

For proposition (3): assume $t \in [\![A]\!]^-$, and show $t \in \mathbf{SN}$. By the definition of $[\![A]\!]^-$ and the fact that $\mathit{Vars} \subseteq [\![A]\!]^{+c}$ by definition of $[\![A]\!]^{+c}$, we have

$$\nu\, y.y \bullet t \in \mathbf{SN}$$

This implies $t \in \mathbf{SN}$ as required.

For proposition (4): assume $t \in [\![A]\!]^{+c}$, and consider the following cases. If $t \in \mathit{Vars}$ or $A \equiv \langle + \rangle$, then $t$ is normal and the result is immediate. So suppose $A \equiv A_1 \to_+ A_2$. Then $t \equiv \lambda x.t'$ for some $x$ and $t'$ where for all $t'' \in [\![A_1]\!]^+$, $[t''/x]t' \in [\![A_2]\!]^+$. By IH(2), the variable $x$ itself is in $[\![A_1]\!]^+$, so we know that $t' \equiv [x/x]t' \in [\![A_2]\!]^+$. Then by IH(1) we have $t' \in \mathbf{SN}$, which implies $\lambda x.t' \in \mathbf{SN}$. If $A \equiv A_1 \to_- A_2$, then $t \equiv \langle t_1, t_2 \rangle$ for some $t_1 \in [\![A_1]\!]^-$ and $t_2 \in [\![A_2]\!]^+$. By IH(3) and IH(1), $t_1 \in \mathbf{SN}$ and $t_2 \in \mathbf{SN}$, which implies $\langle t_1, t_2 \rangle \in \mathbf{SN}$. The cases for $A \equiv A_1 \wedge_p A_2$ are similar to this one.

C.7. **Proof of Theorem 56: Soundness.** The proof is by induction on the derivation of $\Gamma \vdash_c t : p\, A$. We consider the two possible polarities for the conclusion of the typing judgment separately.

Case.

$$\frac{}{\Gamma, x : p\, A, \Gamma' \vdash_c x : p\, A}\ \textsc{ClassAx}$$

Since $\sigma \in [\![\Gamma, x : p\, A, \Gamma']\!]$, $\sigma(x) \in [\![A]\!]^p$ as required.

Case.

$$\frac{}{\Gamma \vdash_c \mathbf{triv} : +\, \langle + \rangle}\ \textsc{ClassUnit}$$

We have $\mathbf{triv} \in [\![\langle + \rangle]\!]^{+c}$ by definition.

Case.

$$\frac{}{\Gamma \vdash_c \mathbf{triv} : -\, \langle - \rangle}\ \textsc{ClassUnit}$$

To prove $\mathbf{triv} \in [\![\langle - \rangle]\!]^-$, it suffices to assume arbitrary $y \in \mathit{Vars}$ and $t \in [\![\langle - \rangle]\!]^{+c}$, and show $\nu\, y.t \bullet \mathbf{triv} \in \mathbf{SN}$. By definition of $[\![\langle - \rangle]\!]^{+c}$, $t \in \mathit{Vars}$, and then $\nu\, y.t \bullet \mathbf{triv}$ is in normal form.

Case.

$$\frac{\Gamma \vdash_c t_1 : +\, A \qquad \Gamma \vdash_c t_2 : +\, B}{\Gamma \vdash_c (t_1, t_2) : +\, A \wedge_+ B}\ \textsc{ClassAnd}$$

By Lemma 55, it suffices to show $(\sigma t_1, \sigma t_2) \in [\![A \wedge_+ B]\!]^{+c}$. This follows directly from the definition of $[\![A \wedge_+ B]\!]^{+c}$, since the IH gives us $\sigma t_1 \in [\![A]\!]^+$ and $\sigma t_2 \in [\![B]\!]^+$.

Case.

$$\frac{\Gamma \vdash_c t_1 : -\, A_1 \qquad \Gamma \vdash_c t_2 : -\, A_2}{\Gamma \vdash_c (t_1, t_2) : -\, A_1 \wedge_- A_2}\ \textsc{ClassAnd}$$

It suffices to assume arbitrary $y \in \mathit{Vars}$ and $t' \in [\![A_1 \wedge_- A_2]\!]^{+c}$, and show $\nu\, y.t' \bullet (\sigma\, t_1, \sigma\, t_2) \in \mathbf{SN}$. If $t' \in \mathit{Vars}$, then this follows by Lemma 53 from the facts that $\sigma t_1 \in [\![A_1]\!]^+$ and $\sigma t_2 \in [\![A_2]\!]^+$, which we have by the IH. So suppose $t'$ is of the form $\mathbf{in}_d\, t''$ for some $d$ and some $t'' \in [\![A_d]\!]^+$. By the definition of $\mathbf{SN}$, it suffices to show that all one-step successors $t_a$ of the term in question are $\mathbf{SN}$. The proof of this is by inner induction on $\delta(t'') + \delta(\sigma t_1) + \delta(\sigma t_2)$, which exists by Lemma 53, using also Lemma 52. Suppose that we step to $t_a$ by stepping $t''$, $\sigma t_1$, or $\sigma t_2$. Then the result holds by the inner IH. So consider the step

$$\nu\, y.\mathbf{in}_d\, t'' \bullet (\sigma\, t_1, \sigma\, t_2) \rightsquigarrow \nu\, y.t'' \bullet \sigma\, t_d$$

We then have $\nu\, y.t'' \bullet \sigma\, t_d \in \mathbf{SN}$ from the facts that $t'' \in [\![A_d]\!]^+$ and $\sigma t_d \in [\![A_d]\!]^-$, by the definition of $[\![A_d]\!]^+$.

Case.

$$\frac{\Gamma \vdash_c t : +\, A_d}{\Gamma \vdash_c \mathbf{in}_d\ t : +\, A_1 \wedge_- A_2}\ \textsc{ClassAndBar}$$

By Lemma 55, it suffices to prove $\mathbf{in}_d\, \sigma\, t \in [\![A_1 \wedge_- A_2]\!]^+$, but by the definition of $[\![A_1 \wedge_- A_2]\!]^+$, this follows directly from $\sigma t \in [\![A_d]\!]^+$, which we have by the IH.

Case.

$$\frac{\Gamma \vdash_c t : -\, A_d}{\Gamma \vdash_c \mathbf{in}_d\ t : -\, A_1 \wedge_+ A_2}\ \textsc{ClassAndBar}$$

To prove $\mathbf{in}_d\, \sigma\, t \in [\![A_1 \wedge_+ A_2]\!]^-$, it suffices to assume arbitrary $y \in \mathit{Vars}$ and $t' \in [\![A_1 \wedge_+ A_2]\!]^{+c}$, and show $\nu\, y.t' \bullet \mathbf{in}_d\, \sigma\, t \in \mathbf{SN}$. If $t' \in \mathit{Vars}$, then this follows from the fact that $\sigma t \in \mathbf{SN}$, which we have by Lemma 53 from $\sigma t \in [\![A_d]\!]^-$ (which the IH gives us). So suppose $t'$ is of the form $(s_1, s_2)$ for some $s_1 \in [\![A_1]\!]^+$ and $s_2 \in [\![A_2]\!]^+$. It suffices to prove that all one-step successors of the term in question are in $\mathbf{SN}$, as we did in a previous case above. Lemma 53 lets us proceed by inner induction on $\delta(\sigma t) + \delta(s_1) + \delta(s_2)$, using also Lemma 52. If we step $\sigma t$, $s_1$ or $s_2$, then the result holds by inner IH. Otherwise, we have the step

$$\nu\, y.(s_1, s_2) \bullet \mathbf{in}_d\, \sigma\, t \rightsquigarrow \nu\, y.s_d \bullet \sigma\, t$$

And this successor is in $\mathbf{SN}$ by the facts that $s_d \in [\![A_d]\!]^+$ and $\sigma t \in [\![A_d]\!]^-$, from the definition of $[\![A_d]\!]^+$.

Case.

$$\frac{\Gamma, x : +\, A \vdash_c t : +\, B}{\Gamma \vdash_c \lambda x.t : +\, A \rightarrow_+ B}\ \textsc{ClassImp}$$

By Lemma 55, it suffices to assume arbitrary $y \in \mathit{Vars}$ and $t' \in [\![A]\!]^+$, and prove $[t'/x](\sigma t) \in [\![B]\!]^+$. But this follows immediately from the IH, since $[t'/x](\sigma t) \equiv (\sigma[x \mapsto t'])t$ and $\sigma[x \mapsto t] \in [\![\Gamma, x : +\, A]\!]$.

Case.

$$\frac{\Gamma, x : -\, A \vdash_c t : -\, B}{\Gamma \vdash_c \lambda x.t : -\, A \rightarrow_- B}\ \textsc{ClassImp}$$

It suffices to assume arbitrary $y \in \mathit{Vars}$ and $t' \in [\![A \rightarrow_- B]\!]^{+c}$, and show $\nu\, y.t' \bullet \lambda x.\sigma\, t \in \mathbf{SN}$. Let us first observe that $\sigma\, t \in \mathbf{SN}$, because by the IH, for all $\sigma' \in [\![\Gamma, x : -\, A]\!]$, we have $\sigma' t \in [\![B]\!]^-$, and $[\![B]\!]^- \subseteq \mathbf{SN}$ by Lemma 53. We may instantiate this with $\sigma[x \mapsto x]$, since by Lemma 53, $x \in [\![A]\!]^-$. Since $\sigma\, t \in \mathbf{SN}$, we also have $\lambda x.\sigma\, t \in \mathbf{SN}$. Now let us consider cases for the assumption $t' \in [\![A \rightarrow_- B]\!]^{+c}$. If $t' \in \mathit{Vars}$ then we directly have $\nu\, y.t' \bullet \lambda x.\sigma\, t \in \mathbf{SN}$ from $\lambda x.\sigma\, t \in \mathbf{SN}$. So assume $t' \equiv \langle t_1, t_2 \rangle$ for some $t_1 \in [\![A]\!]^-$ and $t_2 \in [\![B]\!]^+$. By Lemma 53 again, we may reason by inner induction on $\delta(t_1) + \delta(t_2) + \delta(\sigma t)$ to show that all one-step successors of $\nu\, y.\langle t_1, t_2 \rangle \bullet \lambda x.\sigma\, t$ are in $\mathbf{SN}$, using also Lemma 52. If $t_1$, $t_2$, or $\sigma t$ steps, then the result follows by the inner IH. So suppose we have the step

$$\nu\, y.\langle t_1, t_2 \rangle \bullet \lambda x.\sigma\, t \rightsquigarrow \nu\, y.t_2 \bullet [t_1/x](\sigma\, t)$$

Since $t_1 \in [\![A]\!]^-$, the substitution $\sigma[x \mapsto t_1]$ is in $[\![\Gamma, x : -\, A]\!]$. So we may apply the IH to obtain $[t_1/x](\sigma t) \equiv \sigma[x \mapsto t_1] \in [\![B]\!]^-$. Then since $t_2 \in [\![B]\!]^+$, we have $\nu\, y.t_2 \bullet [t_1/x](\sigma\, t)$ by definition of $[\![B]\!]^+$.

Case.

$$\frac{\Gamma \vdash_c t_1 : -\,A \qquad \Gamma \vdash_c t_2 : +\,B}{\Gamma \vdash_c \langle t_1, t_2\rangle : +\,(A \to_- B)}\ \textsc{ClassImpBar}$$

By Lemma 55, as in previous cases of positive typing, it suffices to prove $\langle \sigma\, t_1, \sigma\, t_2\rangle \in [\![A \to_- B]\!]^{+c}$. By the definition of $[\![A \to_- B]\!]^{+c}$, this follows directly from $\sigma t_1 \in [\![A]\!]^-$ and $\sigma t_2 \in [\![B]\!]^+$, which we have by the IH.

Case.

$$\frac{\Gamma \vdash_c t_1 : +\,A \qquad \Gamma \vdash_c t_2 : -\,B}{\Gamma \vdash_c \langle t_1, t_2\rangle : -\,(A \to_+ B)}\ \textsc{ClassImpBar}$$

It suffices to assume arbitrary $y \in \mathit{Vars}$ and $t' \in [\![A \to_+ B]\!]^{+c}$, and show $\nu\, y.t' \bullet \langle \sigma\, t_1, \sigma\, t_2\rangle \in \mathbf{SN}$. By the IH, we have $\sigma t_1 \in [\![A]\!]^+$ and $\sigma t_2 \in [\![B]\!]^-$, and hence $\sigma t_1 \in \mathbf{SN}$ and $\sigma t_2 \in \mathbf{SN}$ by Lemma 53. If $t' \in \mathit{Vars}$, then these facts are sufficient to show the term in question is in $\mathbf{SN}$. So suppose $t' \equiv \lambda x.t_3$, for some $x \in \mathit{Vars}$ and $t''$ such that for all $t_4 \in [\![A]\!]^+$, $[t_4/x]t_3 \in [\![B]\!]^+$. By similar reasoning as in a previous case, we have $t_3 \in \mathbf{SN}$. So we may proceed by inner induction on $\delta(t_1)+\delta(t_2)+\delta(t_3)$ to show that all one-step successors of $\nu\, y.\lambda x.t_3 \bullet \langle \sigma\, t_1, \sigma\, t_2\rangle$ are in $\mathbf{SN}$, using also Lemma 52. If it is $t_3$, $\sigma t_1$, or $\sigma t_2$ which steps, then the result follows by the inner IH. So consider this step:

$$\nu\, y.\lambda x.t_3 \bullet \langle \sigma\, t_1, \sigma\, t_2\rangle \rightsquigarrow \nu\, y.[\sigma\, t_1/x]t_3 \bullet \sigma\, t_2$$

Since we have that $\sigma t_1 \in [\![A]\!]^+$, the assumption about substitution instances of $t_3$ gives us that $[\sigma t_1/x]t_3 \in [\![B]\!]^+$, which is then sufficient to conclude $\nu\, y.[\sigma\, t_1/x]t_3 \bullet \sigma\, t_2 \in \mathbf{SN}$ by the definition of $[\![B]\!]^+$.

Case.

$$\frac{\Gamma, x : -\,A \vdash_c t_1 : +\,B \qquad \Gamma, x : -\,A \vdash_c t_2 : -\,B}{\Gamma \vdash_c \nu\, x.t_1 \bullet t_2 : +\,A}\ \textsc{ClassCut}$$

It suffices to assume arbitrary $y \in \mathit{Vars}$ and $t' \in [\![A]\!]^-$, and show $\nu\, y.(\nu\, x.\sigma\, t_1 \bullet \sigma\, t_2) \bullet t' \in \mathbf{SN}$. By the IH and part 2 of Lemma 53, we know that $\sigma t_1 \in [\![B]\!]^+$ and $\sigma t_2 \in [\![B]\!]^-$. By Lemma 53 again, we have $t' \in \mathbf{SN}$, $\sigma t_1 \in \mathbf{SN}$, and $\sigma t_2 \in \mathbf{SN}$. So we may reason by induction on $\delta(t') + \delta(\sigma t_1) + \delta(\sigma t_2)$ to show that all one-step successors of $\nu\, y.(\nu\, x.\sigma\, t_1 \bullet \sigma\, t_2) \bullet t'$ are in $\mathbf{SN}$, using also Lemma 52. If it is $t'$, $\sigma t_1$, or $\sigma t_2$ which steps, then the result follows by the inner IH. The only possible other reduction is by the RBetaL reduction rule (Figure 8). And then, since $t' \in [\![A]\!]^-$, we may apply the IH to conclude that $[t'/x](\sigma t_1) \in [\![B]\!]^+$ and $[t'/x](\sigma t_2) \in [\![B]\!]^-$. By the definition of $\in [\![B]\!]^+$, this suffices to prove $\nu\, y.[t'/x]\sigma\, t_1 \bullet [t'/x]\sigma\, t_2 \in \mathbf{SN}$, as required.

Case.

$$\frac{\Gamma, x : -\,A \vdash_c t_1 : +\,B \qquad \Gamma, x : -\,A \vdash_c t_2 : -\,B}{\Gamma \vdash_c \nu\, x.t_1 \bullet t_2 : -\,A}\ \textsc{ClassCut}$$

It suffices to consider arbitrary $y \in \mathit{Vars}$ and $t' \in [\![A]\!]^{+c}$, and show $\nu\, y.t' \bullet (\nu\, x.\sigma\, t_1 \bullet \sigma\, t_2) \in \mathbf{SN}$. By the IH and part 2 of Lemma 53, we have $\sigma t_1 \in [\![B]\!]^+$ and $\sigma t_2 \in [\![B]\!]^-$, which implies $\sigma t_1 \in \mathbf{SN}$ and $\sigma t_2 \in \mathbf{SN}$ by Lemma 53 again. We proceed by inner induction on $\delta(t') + \delta(\sigma t_1) + \delta(\sigma t_2)$, using Lemma 52, to show that all one-step successors of $\nu\, y.t' \bullet (\nu\, x.\sigma\, t_1 \bullet \sigma\, t_2)$ are in $\mathbf{SN}$. If it is $t'$, $\sigma t_1$, or $\sigma t_2$ which steps, then the result holds by inner IH. The only other reduction possible is by RBetaR, since $t'$ cannot be a cut term by the definition of $[\![A]\!]^{+c}$. In this case, the IH gives us $[t'/x]\sigma t_1 \in [\![B]\!]^+$ and $[t'/x]\sigma t_2 \in [\![B]\!]^-$, and we then have $\nu\, y.[t'/x]\sigma\, t_1 \bullet [t'/x]\sigma\, t_2 \in \mathbf{SN}$ by the definition of $[\![B]\!]^+$.